MATH 332: EXAM 1 REVIEW

The following exercises are intended to hit some highlights from the first six weeks of class. They are *not* comprehensive. Parts of two of the problems will appear on the exam. These exercises will not be collected, but you are encouraged to discuss them with your peers and the instructor.

Exercise 1. Let O be the group of rigid symmetries of the regular octahedron. You have already proven that O has order 24. Show that $O \cong S_4$ by considering the action of O on the set of pairs of opposite sides of the octahedron.

Exercise 2. Let C be a normal subgroup of of the group A and let D be a normal subgroup of the group B. Prove that $(C \times D) \trianglelefteq (A \times B)$ and $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.

Exercise 3. Use Lagrange's theorem in the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$ to prove *Euler's theorem*: $a^{\varphi(n)} \equiv 1 \pmod n$ for every integer a relatively prime to n. (Here φ is Euler's totient function: $\varphi(n) = |\{a \in \mathbb{Z} \mid 1 \leqslant a \leqslant n,\ (a,n) = 1\}$.)

Exercise 4. Let G be a finite group and let $\pi : G \to S_G$ be the left regular representation of G. Prove that if x is an element of order n and $|G| = mn$, then $\pi(x)$ is the product of m n-cycles. Deduce that $\pi(x)$ is an odd permutation if and only if $|x|$ is even and $|G|/|x|$ is odd.

Exercise 5. Prove that the center of the symmetric group S_n is trivial for all $n \geqslant 3$. (*Hint*: Remember the cycle decomposition of $\tau\sigma\tau^{-1}$ in terms of the cycle decomposition of σ; handcraft non-commuting elements.)

Exercise 6. If p is prime, prove that there are no simple groups of order $2p$.

Exercise 7. If $|G| = 21$ and G is not cyclic, how many Sylow 3-subgroups does G have?

Exercise 8. Let $\mathbb{Q}^\times$ denote the multiplicative group of nonzero rational numbers. Show that $\mathbb{Q}^\times_+ = \{q \in \mathbb{Q}^\times \mid q > 0\}$ is a subgroup of $\mathbb{Q}^\times$. Prove that

$$\mathbb{Q}^\times \cong \mathbb{Q}^\times_+ \times \{\pm 1\}.$$

Can you come up with an analogue of this result when the group is $\mathbb{R}^\times$? $\mathbb{C}^\times$?

Exercise 9. Find, with proof, the commutator subgroup of S_n, $n \geqslant 3$.

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