MATH 332: EXAM 1

Name: ______________________________

Instructions. Take-home exam. 120 minutes. Closed everything except blank scratch paper and one two-sided 8.5″×11″ page of notes.

Do not read beyond this page until you are seated in a distraction-free environment. Once you read beyond this page, your time has started and you have 120 minutes to complete the exam. All of the work must be your own and you may not rely on any outside resources. *The Honor Principle prevails.*

Please write your name in the space provided above. Please provide solutions (as cleanly presented as possible) on the front side of each page. The back side of each sheet of paper is intended for scratch work and **will not be graded** unless there is an explicit note directing the instructor to do so.

Once your 120 minutes are up, scan your work and email it to `ormsbyk@reed.edu` with subject line Math 332 Exam 1. (Please first send the scan to yourself, check that the pdf appropriately reflects your work (*i.e.*, is legible and contains all pages), rename the file along the lines of `MyNameExam1.pdf`, and then follow the above email instructions.) **Save the paper copy of your exam** until you have received your graded copy of Exam 1!

*Your scan is due by 11:59*PM *on Monday, March 9. Remember that we just started daylight savings time!*

Problem	Points	Out of
1		20
2		15
3		20
4		25
5		20
6		20
Total		120

Date: 9.III.15.

Problem 1 (points). Let G and H be groups, let n be a positive integer, let Z_n denote the cyclic group of order n, and let $(\mathbb{Z}/n\mathbb{Z})^\times$ be the multiplicative group $\{a \in \mathbb{Z}/n\mathbb{Z} \mid \exists b \in \mathbb{Z}/n\mathbb{Z} \text{ such that } a \cdot b = 1\}$. For the following statements, circle T if the statement is true and F if the statement is false. You do not need to show any work.

If $\varphi : G \to H$ is a homomorphism and $J \leqslant H$ is a subgroup, then $\varphi^{-1}(J)$ is a subgroup of G.	T or F
If $x \in G$ has order n and $S = \langle x \rangle$ is the subgroup of G generated by x, then $\lvert S \rvert = n$.	T or F
If $H \leqslant G$, then $\lvert G \rvert = \lvert H \rvert \cdot [G : H]$.	T or F
For any $x \in G$, $\langle x \rangle$ is a normal subgroup of G.	T or F
Let p be a prime number dividing the order of G and the order of H. If Sylow p-subgroups of G have order p^α and Sylow p-subgroups of H have order p^β, then Sylow p-subgroups of $G \times H$ have order $p^{\alpha+\beta}$.	T or F
$Q_8/\{\pm 1\} \cong Z_4$.	T or F
$\mathbb{Z}/n\mathbb{Z} \cong Z_n$.	T or F
The group of automorphisms of Z_n is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$.	T or F
There exists an action $S_5 \circlearrowright A$ of the symmetric group on five letters S_5 on a set A such that there exist $a, b \in A$ such that $S_5 \cdot a \cap S_5 \cdot b \neq \varnothing$ and $S_5 \cdot a \neq S_5 \cdot b$.	T or F
If $x, y \in G$ are elements of order m and n, respectively, then $\lvert \langle x, y \rangle \rvert = \gcd(m, n)$.	T or F

Problem 2 (15 points). Let G be a group. Prove that if H is a subgroup of $Z(G)$, the center of G, then H is a normal subgroup of G.

Problem 3 (20 points). If p is prime, prove that there are no simple groups of order $2p$.

Problem 4 (25 points). Let R denote the set of polynomials in variables x_1, x_2, x_3, x_4 with integer coefficients.

(a) Verify that the operation $\cdot : S_4 \times R \to R$ defined by
$$\sigma \cdot f(x_1, x_2, x_3, x_4) = f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$$
is a (left) action of S_4 on R.

(b) Compute the orbit of the polynomial $x_1x_2 + x_3x_4$ under this action.

(c) Use the orbit-stabilizer theorem to determine the size of the stabilizer of $x_1x_2 + x_3x_4$.

(d) Compute the stabilizer of $x_1x_2 + x_3x_4$ under this action and check your answer from part (c).

Problem 5 (20 points). Let C be a normal subgroup of of the group A and let D be a normal subgroup of the group B. Prove that $(C \times D) \trianglelefteq (A \times B)$ and $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.

Problem 6 (20 points). Compute, with proof, the number of homomorphisms from A_4 to Z_{12}. (*Hint*: You may want to start by computing the abelianization of A_4.) What does this tell you about cyclic quotients of A_4?