

a. $P = 2x^2 - 3y^3zx + 4y^2z^5$ ✓ $q = (5y^3z^4 - 3z^3 + 7)x^2$ ✓

$\bar{P} = 2x^2 + y^2z^5$ ✓ $\bar{q} = (2y^3z^4 + 1)x^2$ ✓

b. degree of P is 7, degree of q is 9, degree of \bar{P} is 7, degree of \bar{q} is 9

degree of P in x is 2	degree of q in x is 2	degree of \bar{P} in x is 2
degree of P in y is 3	degree of q in y is 3	degree of \bar{P} in y is 2
degree of P in z is 5	degree of q in z is 4	degree of \bar{P} in z is 5

d. $PQ = (2x^2 - 3y^3zx + 4y^2z^5)(5y^3z^4x^2 - 3z^3x^2 + 7x^2) = 10y^3z^4x^4 - 3z^3x^4 + 14x^4 - 15y^6z^5x^3$
 $+ 9y^3z^4x^3 - 21y^3z^3x^3 + 20y^5z^9x^2 - 12y^2z^8x^2 + 28y^3z^5x^2$
degree of PQ in x is 4
degree of PQ in y is 6
degree of PQ in z is 9

$\bar{P}\bar{q} = (2x^2 + y^2z^5)(2y^3z^4x^2 + x^2) = x^4y^3z^4 + 2x^4 + 2x^2y^5z^9 + x^2y^2z^5$
degree of $\bar{P}\bar{q}$ in x is 4
degree of $\bar{P}\bar{q}$ in y is 5
degree of $\bar{P}\bar{q}$ in z is 9.

e. $PQ = 20y^5x^2z^9 - 12y^3x^2z^8 + (28y^3x^2 - 15y^6x^3)z^5 + (10y^3x^4 + 9y^3x^3)z^4 - 3x^4z^7 - 21y^3x^3z + 14x^2$

$\bar{P}\bar{q} = 2x^2y^5z^4 + x^2y^2z^5 + x^4y^3z^4 + 2x^4$

2. Observe that $c \in (x)$ $c = \sum_{i=1}^n c_i x^i$ where $c_i \in \mathbb{Q}[y]$. ✓

(5) So if $c = ab$, for $a \in \mathbb{Q}[x,y]$, $b \in \mathbb{Q}[x,y]$, then $a \circ b$ must be 0. However, \mathbb{Q} is an integral domain, which means that $\mathbb{Q}[y]$ is an integral domain. Therefore, w.l.o.g. say that $a_0 = 0$. Then

$$a = \sum_{i=1}^m a_i x^i \in (x)$$

Thus (x) is prime. good.

Then if $c \in (x,y)$, $c = \sum_{i=0}^n c_i x^i$, ~~$c_i \in \mathbb{Q}[y]$~~ . ~~Suppose $c = ab$.~~
 ~~$c_i \in \mathbb{Q}[y]$ if $i \geq 1$, $c_0 \in y\mathbb{Q}[y]$.~~

Then if $a \circ b = c$, $a \circ b \in y\mathbb{Q}[y]$. Then for $a_0, b_0 \in \mathbb{Q}[y]$,

$$a_0 = \sum_{j=0}^n a_j y^j, \quad b_0 = \sum_{k=0}^m b_k y^k. \quad a_j, b_k \in \mathbb{Q}$$

$a_0 b_0 \in y\mathbb{Q}[y]$ implies that $a_0 b_0 = 0$. \mathbb{Q} is an integral domain, so w.l.o.g. say $a_0 = 0$. Then $a_0 \in y\mathbb{Q}[y]$, which means that $a \in (x,y)$. Thus (x,y) is prime as well. ✓

(x) is the set of all elements with x as a factor of all terms.

(x,y) is the set of all elements with x or y as a factor of all terms.

Thus $(x) \subsetneq (x,y)$, so unless $(x,y) = \mathbb{Q}[x,y]$ (which it doesn't) (x) cannot be maximal.

(x,y) cannot be equal to $\mathbb{Q}[x,y]$ because it does not include ~~constant~~ polynomials with non-zero constant terms. i.e. doesn't contain constants.

Observe that $\mathbb{Q}[x,y]/(x,y) \cong \mathbb{Q}$. (x,y) is the set of all elements in $\mathbb{Q}[x,y]$ without a constant term. Thus for $c \in \mathbb{Q}[x,y]$ $c + (x,y) = c_0 + (x,y)$, where c_0 is the constant term in \mathbb{Q} . We know that \mathbb{Q} is a field. Thus (x,y) is maximal. ✓

Problem 3. Prove that a polynomial ring in infinitely many variables

$$R[x_1, x_2, x_3, \dots]$$

where R is any commutative ring (with $1 \neq 0$) contains ideals which are not finitely generated. (Thus $R[x_1, x_2, x_3, \dots]$ is an example of a *non-Noetherian ring*.)

Answer:

Consider the ideal generated by the infinite set (x_1, x_2, \dots) , or the ideal generated by the infinite set of first degree monic polynomials of each variable. If this ideal were finitely generated, then we would have to have a variable x_i be in the ideal generated by a finite ideal (x_1, \dots, x_n) . However, since no variables equal a linear combination of any of the other variables in our polynomial ring, no such finitely generated ideal exists. Thus, (x_1, x_2, \dots) is non-finitely generated, proving that $R[x_1, x_2, x_3, \dots]$ is an example of a non-Noetherian ring.

+ $I \subseteq R$ be a homogeneous ideal, then it's obvious that the set of all A is
ideal of I. Then it's clear that the homogeneous parts of A generate I. ✓ (\Leftarrow)

other hand (\Rightarrow), suppose that I is an ideal generated by the homogeneous polynomials

and let $g \in I$ be a polynomial, then $g = \sum_{i=1}^k a_i f_i$, so that the homogeneous part

term in the sum is also in I, thus I is homogeneous. ✓

problem 5. F a field, $f(x) \in F[x]$.

$F[x]$ is a Euclidean domain, thus a PID. \Rightarrow max'l ideals are principle ideals
generated by irreducibles $\Rightarrow F[x]/(f(x))$ is a field iff $f(x)$ is irreducible

6. (a) $\mathbb{Z}[x]/(2) = (\mathbb{Z}/2\mathbb{Z})[x]$. In other words, it contains polynomials in x with binary coefficients.

(b) (x) is the set of polynomials with 0 constant term. Therefore, $\mathbb{Z}[x]/(x)$ is just the constant terms. Consequently, it looks exactly the same as \mathbb{Z} .

(c) (x^2) is the set of polynomials in x with 0 constant and first degree terms. Therefore, $\mathbb{Z}[x]/(x^2)$ looks like $\{a + bx \mid a, b \in \mathbb{Z}\}$.

(d) $(x^2, y^2, 2)$ is the set of all polynomials with even coefficients and polynomials with degree in x or y greater than or equal to 2. Consequently, elements of $I = \mathbb{Z}/(x^2, y^2, 2)$ are of the form $a + bx + cy + dxy$ where $a, b, c, d \in \mathbb{Z}/2\mathbb{Z}$. If $x = \bar{1}$, then $x^2 = 1$. If $x \neq \bar{1}, \bar{0}$, then $x = a + bx + cy + dxy + (x^2, y^2, 2)$ where either $b, c, d \neq 0$. Well,

$$(a + bx + cy + dxy + (x^2, y^2, 2))^2 = a^2 + b^2x^2 + c^2y^2 + d^2x^2y^2 + \text{other terms}$$

None of the other terms will have equivalent multidimensional order to these terms, and consequently will not affect the following analysis. By our assumption, either b, c or d is 1. Therefore, there will be a non-zero term for something of degree two. Consequently, $x^2 = 0$.

Problem 7. Let F be a field and let R be the set of polynomials in $F[x]$ whose coefficient of x is 0. Prove that R is a subring of $F[x]$. Use the equation $x^6 = (x^2)^3 = (x^3)^2$ to prove that R is not a UFD.

S R is closed under addition because $0x + 0x = 0$, under multiplication because $(a_0 + a_2x^2 + \dots)(b_0 + b_2x^2 + \dots) = a_0b_0 + (a_0b_2 + a_2b_0)x^2 + \dots$ and contains the identity elements 0 and 1 and additive inverses $-(a_0 + a_2x^2 + \dots) = (-a_0) + (-a_2)x^2 + \dots$. $x^3 \cdot x^3 = x^6 = x^2 \cdot x^2 \cdot x^2$ and as x^2 and x^3 are both irreducible elements in R , R is not a UFD. ✓

Exercise 8. Determine with proof whether the polynomials are irreducible in the rings below.

(a) $x^2 + x + 1$ in $(\mathbb{Z}/2\mathbb{Z})[x]$

This is irreducible as it does not have a root.

(b) $x^3 + x + 1$ in $(\mathbb{Z}/3\mathbb{Z})[x]$

✓ Consider $(x+2)(x^2+x+2) = x^3 + 3x^2 + 4x + 4$
 $= x^3 + x + 1$

✓ Thus $x^3 + x + 1$ is red. in $(\mathbb{Z}/3\mathbb{Z})[x]$

Also note $P(1) = 0$,

(c) $x^4 + 1$ in $(\mathbb{Z}/5\mathbb{Z})[x]$

✓ In $(\mathbb{Z}/5\mathbb{Z})$, $-1 \equiv -4$ so $x^4 + 1 \equiv 5(x^2 - 2)(x^2 + 2)$
 $\equiv -5(x^3 + 3)(x^2 + 2)$

(d) $x^4 + 10x^2 + 1$ in $\mathbb{Z}[x]$

✓ As $x^4 + 10x^2 + 1$ has no roots in \mathbb{Z} then we know that if $x^4 + 10x^2 + 1$ factors its factors take the form $(ax^2 + bx + c)$

Solving for a, b, c , a', b', c' ,

$$\begin{aligned} x^4 + 10x^2 + 1 &= (ax^2 + bx + c)(a'x^2 + b'x + c') \\ &= aa'x^4 + ab'x^3 + ac'x^2 \\ &\quad + a'b'x^3 + bb'x^2 + bc'x \\ &\quad + a'c'x^2 + b'cx + cc' \end{aligned}$$

We see $a = a' = \pm 1$ & $c = c' = \pm 1$

but $a'c' + ac' = 10$, a contradiction so

$x^4 + 10x^2 + 1$ is irreducible in $\mathbb{Z}[x]$.

or use Eisenstein?

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Problem 9. Show that the polynomial $(x-1)(x-2)\cdots(x-n)+1$ is irreducible in $\mathbb{Z}[x]$ for all $n \geq 1, n \neq 4$.

Answer:

Suppose $f(x) = (x-1)(x-2)\cdots(x-n)+1 = g(x)h(x)$ for some $g(x), h(x) \in \mathbb{Z}[x]$, where $\deg(g), \deg(h) < n$. Note that for $1 \leq \alpha \leq n$, $f(\alpha) = 1 = g(\alpha)h(\alpha)$. Thus, $g(\alpha) = h(\alpha) = \pm 1$. Therefore, $g(\alpha) - h(\alpha) = 0$. Since $g(x)$ and $h(x)$ are polynomials of degree less than n , $g(x) - h(x)$ is a polynomial of degree less than n , which has at least n roots. This can only be the case if $g(x) - h(x) = 0$, implying $g(x) = h(x)$. Thus, $f(x) = g(x)^2$, implying $g(\alpha) = \pm 1$ for $1 \leq \alpha \leq n$. Note therefore that n cannot be odd, as $\deg(g(x)^2) = \deg(g) + \deg(g) = 2\deg(g)$ which is even. For $n = 2$, note that this would require $f(x) = x^2 - 3x + 3$. By Eisenstein's criterion, since $f(x)$ is monic and 3 divides both the x and constant coefficients, but 3^2 does not divide 3, $f(x)$ is irreducible. Now suppose $n \geq 6$ implying $k \geq 3$, and let $\deg(f) = 2k$, implying $\deg(g) = k$, for $k \geq 3$. Note that if the function $g(x) - 1$, which is a degree k function, has less than k roots, then

(S)

Define $\varphi: R[x] \rightarrow \mathbb{C}$ by $\varphi(a+bx) = a+b i$. φ is a surjective ring homomorphism.
Also, $\varphi(1+x^2) = 1+i^2 = 1-1 = 0 \Rightarrow 1+x^2 \in \ker \varphi$. Since x^2+1 is irreducible over $R[x]$,

$\mathbb{C}[x]/(x^2+1)$ is a field, so we know that $\ker \varphi = x^2+1$, so by the first isomorphism

theorem $\mathbb{C} \cong R[x]/(x^2+1)$. \square

5 Let $F_{11} = \mathbb{Z}/11\mathbb{Z}$ and let $K_1 = F_{11}[x]/(x^2+1)$ & $K_2 = F_{11}[y]/(y^2+2y+2)$.
 Since we're going to conclude that ~~K_1 & K_2 are isomorphic, it suffices to show that K_1 has 121 elements.~~

Now $\alpha \in K_1$ is of the form : $\alpha = a + bx + (x^2+1)$ where $a, b \in \mathbb{Z}/11\mathbb{Z}, \mathbb{Z}/11\mathbb{Z}$ as 11 elements, so there are $11^2 = 121$ pairings of a, b to define α . Thus $|K_1|$ has exactly 121 elements.

K_1 & K_2 are also fields since $(x^2+1), (y^2+2y+2)$ have no roots in F_{11} and are therefore reducible. Similarly, non-elements in K_2 are of the form $a+by + (y^2+2y+2)$, so $|K_2| = |K_1| = 121$.

Let

$$+\quad \psi: K_1 \rightarrow K_2 \text{ by } \begin{aligned} \psi(p(x)) &= p(\bar{y}+1) \\ a+bx+(x^2+1) &\mapsto a+bx+(x^2+2y+2) \end{aligned}$$

$\vdash \exists \alpha, \beta \text{ s.t. } \psi(\alpha) = \psi(\beta)$, then $a_\alpha + b_\alpha x + (x^2+1) = a_\beta + b_\beta x + (x^2+2y+2)$

$\Rightarrow a_\alpha = a_\beta \text{ & } b_\alpha = b_\beta$, thus, $\alpha = \beta$, so ψ is indeed well defined and injective.

\vdash It is clear that ψ is a homomorphism and since $|K_1| = |K_2|$ it is also onto, thus,

ψ is an isomorphism. \square

good.