MATH 332: HOMEWORK 9

Exercise 1. Let $p(x, y, z) = 2x^2 - 3xy^3z + 4y^2z^5$ and $q(x, y, z) = 7x^2 + 5x^2y^3z^4 - 3x^2z^3$ be polynomials in $\mathbb{Z}[x, y, z]$. Let $\bar{p}(x, y, z)$ and $\bar{q}(x, y, z)$ denote the images of $p(x)$ and $q(x)$ in $(\mathbb{Z}/3\mathbb{Z})[x]$ under the canonical reduction map.

(a) Write each of p, q, $\bar{p}$, $\bar{q}$ as a polynomial in x with coefficients in $\mathbb{Z}[y, z]$ or $(\mathbb{Z}/3\mathbb{Z})[y, z]$.
(b) Find the degree of p, q, $\bar{p}$, $\bar{q}$.
(c) Find the degree of p, q, $\bar{p}$, $\bar{q}$ in each of the variables x, y, z.
(d) Compute pq, $\bar{p}\bar{q}$ and find the degree of each in each of the variables x, y, z.
(e) Write pq, $\bar{p}\bar{q}$ as a polynomial in z with coefficients in $\mathbb{Z}[x, y]$ or $(\mathbb{Z}/3\mathbb{Z})[x, y]$.

Exercise 2. Prove that the ideals (x) and (x, y) are prime in $\mathbb{Q}[x, y]$, but only (x, y) is maximal.

Problem 3. Prove that a polynomial ring in infinitely many variables

$$R[x_1, x_2, x_3, \ldots]$$

where R is any commutative ring (with $1 \neq 0$) contains ideals which are not finitely generated. (Thus $R[x_1, x_2, x_3, \ldots]$ is an example of a *non-Noetherian ring*.)

Challenge 4. An ideal I in $R[x_1, \ldots, x_n]$ is called a *homogeneous ideal* if whenever $p(x) \in I$, then each of the homogeneous components of p is also in I. Prove that an ideal is a homogeneous ideal if and only if it may be generated by homogeneous polynomials.

Problem 5. Let $f(x)$ be a polynomial in $F[x]$, F a field. Prove that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible.

Problem 6. Briefly describe the ring structure of each of the following rings:
(a) $\mathbb{Z}[x]/(2)$,
(b) $\mathbb{Z}[x]/(x)$,
(c) $\mathbb{Z}[x]/(x^2)$,
(d) $\mathbb{Z}[x, y]/(x^2, y^2, 2)$.

Show that $\alpha^2 = 0$ or 1 for every α in the last ring and determine those elements with $\alpha^2 = 0$.

Problem 7. Let F be a field and let R be the set of polynomials in $F[x]$ whose coefficient of x is 0. Prove that R is a subring of $F[x]$. Use the equation $x^6 = (x^2)^3 = (x^3)^2$ to prove that R is not a UFD.

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Exercise 8. Determine (with proof) whether the following polynomials are irreducible in the rings indicated.

(a) $x^2 + x + 1$ in $(\mathbb{Z}/2\mathbb{Z})[x]$
(b) $x^3 + x + 1$ in $(\mathbb{Z}/3\mathbb{Z})[x]$
(c) $x^4 + 1$ in $(\mathbb{Z}/5\mathbb{Z})[x]$
(d) $x^4 + 10x^2 + 1$ in $\mathbb{Z}[x]$

Problem 9. Show that the polynomial $(x-1)(x-2)\cdots(x-n)+1$ is irreducible in $\mathbb{Z}[x]$ for all $n \geqslant 1$, $n \neq 4$.

Problem 10. Prove that $\mathbb{R}[x]/(x^2+1)$ is a field isomorphic to $\mathbb{C}$.

Problem 11. Let $\mathbb{F}_{11} = \mathbb{Z}/11\mathbb{Z}$. Prove that $K_1 = \mathbb{F}_{11}[x]/(x^2+1)$ and $K_2 = \mathbb{F}_{11}[y]/(y^2+2y+2)$ are both fields with 121 elements. Prove that the map which sends $p(\bar{x}) \in K_1$ to $p(\bar{y}+1) \in K_2$ is well-defined and gives a field isomorphism $K_1 \to K_2$.