MATH 332: HOMEWORK 7

Exercise 1. Let R be a ring with 1. Prove that $(-1)^2 = 1 \in R$ and that if u is a unit in R, then $-u$ is also a unit in R.

Exercise 2. Which of the following are subrings of $\mathbb{Q}$:

(a) the set of rational numbers with odd denominators (when written in lowest terms),
(b) the set of rational numbers with even denominators (when written in lowest terms),
(c) the set of nonnegative rational numbers,
(d) the set of squares of rational numbers,
(e) the set of all rational numbers with odd numerators (when written in lowest terms).

Problem 3. An element a of a ring R is called *idempotent* if $a^2 = a$. A ring R is called *Boolean* if every element of R is idempotent. Prove that Boolean rings are commutative.

Exercise 4. Let K be a field. A *discrete valuation* on K is a function $v : K^\times \to \mathbb{Z}$ satisfying

(i) $v(ab) = v(a) + v(b)$ for all $a, b \in K^\times$ (*i.e.*, v is a homomorphism [think logarithm!]),
(ii) v is surjective, and
(iii) $v(x + y) \geqslant \min\{v(x), v(y)\}$ for all $x, y \in K^\times$ with $x + y \neq 0$.

The set $\mathcal{O}_v = \{x \in K^\times \mid v(x) \geqslant 0\} \cup \{0\}$ is called the *valuation ring* of v.

(a) Prove that $\mathcal{O}_v$ is a subring of K containing 1.
(b) Prove that for each $x \in K^\times$, x or x^{-1} is in $\mathcal{O}_v$.
(c) Prove that an element x is a unit of $\mathcal{O}_v$ if and only if $v(x) = 0$.

Problem 5. Fix a prime p and define $v_p : \mathbb{Q}^\times \to \mathbb{Z}$ by $v_p(a/b) = \alpha$ where $a/b = p^\alpha \cdot c/d$ where $p \nmid c$ and $p \nmid d$. Prove that v_p is a valuation, then prove that

$$\mathcal{O}_{v_p} = \{a/b \in \mathbb{Q} \mid (p, b) = 1\}.$$

Finally, determine exactly what rational numbers constitute $\mathcal{O}_{v_p}^\times$, the units in $\mathcal{O}_{v_p}$.

Remark. The ring $\mathcal{O}_{v_p}$ above is frequently called $\mathbb{Z}_{(p)}$, the ring of *p-local integers*. Look at exercises 3 and 6 on p.238 of the book for another interesting example of a valuation and valuation ring.

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Problem 6. Let $G = \{g_1, g_2, \ldots, g_n\}$ be a finite group. Define the element $N = g_1 + g_2 + \cdots + g_n$, an element of the group ring $\mathbb{Z}G$. Prove that N is in the center of $\mathbb{Z}G$.

Challenge 7. Prove that the rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are not isomorphic.

Exercise 8. Decide which of the following are ideals of the ring $\mathbb{Z}[x]$:
(a) the set of all polynomials whose constant term is a multiple of 3,
(b) the set of all polynomials whose coefficient of x^2 is a multiple of 3,
(c) the set of all polynomials whose constant term, coefficient of x, and coefficient of x^2 are zero,
(d) $\mathbb{Z}[x^2]$, the set of polynomials in which only even powers of x appear,
(e) the set of polynomials whose coefficients sum to 0,
(f) the set of polynomials $p(x)$ such that $p'(0) = 0$, where $p'(x)$ is the usual first derivative of $p(x)$ with respect to x.

Problem 9. Find all ring homomorphisms $\mathbb{Z} \to \mathbb{Z}/30\mathbb{Z}$. In each case describe the kernel and the image.

Challenge 10. Let I and J be ideals of R.
(a) Prove that $I + J$ is the smallest ideal of R containing both I and J.
(b) Prove that IJ is an ideal contained in $I \cap J$.
(c) Give an example where $IJ \neq I \cap J$.
(d) Prove that if R is commutative and if $I + J = R$, then $IJ = I \cap J$.

Problem 11. Let R be a commutative ring with 1. Prove that the principal ideal generated by x in the polynomial ring $R[x]$ is a prime ideal if and only if R is an integral domain. Prove that (x) is a maximal ideal if and only if R is a field.