

1. Let us consider S_4 with subgroups

$A_4 \trianglelefteq S_4$ & $K = \langle (12) \rangle \leq S_4$. As $K \cap A_4 = 1$,

Let $\ell: K \rightarrow \text{Aut}(A_4)$ be defined by mapping
 $k \in K$ to the Automorphism of left conjugation
by k on H . As $|A_{4\#}| | |K| = 24$, $A_{4\#} \times K \cong S_4$

as $A_{4\#} \times K = A_{4\#} K \leq S_4$.

It's also isomorphic to $\text{Aut}(Q_8)$! good.

3. Let (G, U, D) be defined as above, then ~~then~~ except with $n=2$, so that
 $J = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid a \in F \right\}$ and $D = \left\{ \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix} \mid c, b \neq 0 \text{ & } c, b \in F \right\}$. Let $A \in U, B \in D$

Note $B \in U$ by ^{conjugation} $BAB^{-1} = \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix}^{-1} = \begin{bmatrix} 1 & cab^{-1} \\ 0 & 1 \end{bmatrix}$. So we have

the homomorphism $\varphi: F^\times \times F^\times \rightarrow \text{Aut}(F)$ by

$$\varphi((c, b)(a)) = cab^{-1}$$

4) We know from a previous exercise that
 $H = \langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \rangle$ is a unique Sylow 2 subgroup,
 hence normal, and $H \cong \mathbb{Z}_3$. Furthermore,
 (5) $|\frac{SL_2(\mathbb{F}_3)}{H}| = \frac{120}{18} = 13$, meaning the quotient
 group is isomorphic to the cyclic group \mathbb{Z}_3 .
 Thus, we can pick an element in $SL_2(\mathbb{F}_3)$, x ,
 such that xH generates the above quotient
 group. That is, $\langle x \rangle H = G$. Since $H \trianglelefteq G$
 and $H \cap \langle x \rangle = 1$, $G \cong H \times \langle x \rangle$, which by
 the universal property of products is isomorphic
 with $\mathbb{Z}_3 \times \mathbb{Z}_3$. Sensational!

5) Let F_1 and F_2 be free groups of the same finite free rank. This means the underlying sets of F_1 and F_2 are of equal cardinality n .

Let S be a set whose cardinality is n .
Then there exists a set map which takes

Usually n denotes a finite integer.
For you
containing
that the
underlying set
of F_1 is
finite, 2

$s \mapsto F_1(s)$. There also exists a set map which takes $s \mapsto F_2(s)$. By the universal property of free groups, there must exist a unique homomorphism $\varphi: F_1(s) \rightarrow F_2(s)$ and similarly a unique homomorphism $\bar{\vartheta}: F_2(s) \rightarrow F_1(s)$, as shown in the diagram below.

$$\begin{array}{ccc} s & \xrightarrow{\quad} & F_1(s) \\ & \searrow \bar{\vartheta} \uparrow \varphi & \swarrow \\ & F_2(s) & \end{array}$$

Note however that $s \mapsto F_1(s)$ is the function which maps $s \in S$ to the reduced word $s \in F(s)$, and similarly for $s \mapsto F_2(s)$. Thus,

$(s \mapsto F_1(s))_{(s)} = \bar{\vartheta}(s \mapsto F_2(s)) = \bar{\vartheta}(s) = s \in F_1$? Thus, $\bar{\vartheta}$ is the identity mapping $F_2(s) \rightarrow F_1(s)$. Since the identity function is a bijective function, $\bar{\vartheta}$ has an inverse homomorphism, which by uniqueness must equal φ . Since the group mapping

$\bar{\vartheta}: F_2(s) \rightarrow F_1(s)$ is a bijective homomorphism,

$F_1(s) \cong F_2(s)$. Since s was general,

$F_1 \cong F_2$. Note this proof made no use of the fact

$|S| < \infty$, thus the proof has demonstrated the infinite case as well.

Question 6. Give a presentation for A_4 .

(5)

Answer. We know that A_4 is generated by an element of order 2 and an element for order 3. Let $a = (1, 2)(3, 4)$ and $b = (123)$. I know show that $A_4 = \langle a, b \mid a^2 = b^3 = (ab)^3 = 1 \rangle$ by showing that there are 12 words that are irreducible given the relations.

- 1 word of length 0: the identity. ✓
- 2 words of length 1: $\{a, b\}$ ✓
- 3 words of length 2. There are 4 possible: $\{a^2, ab, ba, b^2\}$. But only 3 are irreducible: $\{ab, ba, b^2\}$. ✓
- 4 words of length 3. There are 6 possible: $\{aba, ab^2, ba^2, bab, b^2ab^3\}$. But only 4 are irreducible: $\{aba, ab^2, bab, b^2a\}$. ✓
- 2 words of length 4. There are 8 possible: $\{gba^2, abab, ab^2a, ab^3, baba, bab^2, b^2a^2, a^2ab\}$. But only 2 are irreducible: $\{bab^2, b^2aba\}$. ✓
- 0 words of length 5. All words of length five reduce. ✓

So there are $1 + 2 + 3 + 4 + 2 = 12$ irreducible words, and $|A_4| = 12$, so $\langle a, b \mid a^2 = b^3 = (ab)^3 = 1 \rangle$ is a valid presentation.

perfect.

There is a homomorphism
 $\langle a, b \mid \dots \rangle \rightarrow A_4$ bc your a, b
satisfy the
relation, this
is surjective
by your
statement about
 A_4 being generated
by $(1, 2)(3, 4)$
 $\text{and } (1, 2, 3)$. Your
counting argument
then gives
bijection.

7) Let S be a set where $|S|=n$ for some $n \in \mathbb{N}^+$ and A be an abelian group such that $|A| \geq n$. By the universal property of free groups, there exists a unique homomorphism $\varphi: F(S) \rightarrow A$ such that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\quad} & F(S) \\ & \searrow \varphi & \\ & & A \end{array}$$

Furthermore, we have $\mathbb{Z}(S) = \langle S | [a, b] = 1 \text{ for all } a, b \in S \rangle$. As R for $\mathbb{Z}(S)$ is the set of words $a^{-1}b^{-1}ab$ for all $a, b \in S$ or the commutator subgroup for $F(S)$, the smallest normal subgroup containing R in $F(S)$ must be R itself. Thus, $\mathbb{Z}(S) = F(S)/R$, where R is the commutator subgroup of $F(S)$. Since there exists a homomorphism $\varphi: F(S) \rightarrow A$.

a homomorphism $\text{red}: F(S) \rightarrow F(S)/R = \mathbb{Z}(S)$, and

for $a^{-1}b^{-1}ab \in R$, $\varphi(a^{-1}b^{-1}ab) = \varphi(a)^{-1}\varphi(b)^{-1}\varphi(a)\varphi(b) = 1$,

$R \subseteq \text{ker}(\varphi)$ and by the universal property of quotients, there exists a unique homomorphism

ψ such that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\quad} & F(S) \xrightarrow{\text{red}} \mathbb{Z}(S) \\ & \searrow & \downarrow \psi \\ & & A \end{array}$$

Thus, given a set mapping $S \rightarrow A$, there exists a unique homomorphism $\psi: \mathbb{Z}(S) \rightarrow A$ such that the following diagram commutes.

$$\begin{array}{ccc} S & \xrightarrow{\quad} & \mathbb{Z}(S) \\ & \searrow \psi & \\ & & A \end{array}$$

So, "given an abelian group, and a set map $S \rightarrow A$ there"

$\mathbb{Z}(\{1, 2, \dots, n\}) \cong \mathbb{Z}^n$, first define

set mapping $\pi: \{1, 2, \dots, n\} \rightarrow \mathbb{Z}^n$ by

$i \mapsto e_i$, where e_i is the i^{th} standard basis vector for \mathbb{Z}^n . Note that \mathbb{Z}^n is an

abelian group since $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n)$

$$= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) = (y_1, y_2, \dots, y_n) + (x_1, x_2, \dots, x_n).$$

By the universal property of free abelian groups, there exists a unique homomorphism.

$\varphi: \mathbb{Z}(\{1, 2, \dots, n\}) \rightarrow \mathbb{Z}^n$ such that the following diagram commutes:

$$\begin{array}{ccc} \{1, 2, \dots, n\} & \longrightarrow & \mathbb{Z}(\{1, 2, \dots, n\}) \\ \downarrow \varphi & & \swarrow \text{(might have overlooked a better method by which to use the universal property here)} \\ \mathbb{Z}^n & & \end{array}$$

Furthermore, $\pi(i) = \varphi(i) = e_i$. Thus, $\varphi: \mathbb{Z}(\{1, 2, \dots, n\}) \rightarrow \mathbb{Z}^n$

by $i \mapsto e_i$ is a homomorphism. For i, j as distinct

letters in $\mathbb{Z}(\{1, 2, \dots, n\})$, $\varphi(i) = e_i \neq e_j = \varphi(j)$. Since $\mathbb{Z}(\{1, 2, \dots, n\})$ is abelian, any word can be written in the form $x_1^{x_1} x_2^{x_2} x_3^{x_3} \dots x_n^{x_n}$, φ applied to such a words

in $x_1 \varphi(1) + x_2 \varphi(2) + \dots + x_n \varphi(n) = (x_1, x_2, \dots, x_n)$. Thus, the

only way for $\varphi(w) = (0, 0, \dots, 0)$ is if $w = 0$. Thus,

the kernel of φ is trivial, implying φ is injective.

As shown above, for any element $(x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$,

there exists a word w such that $\varphi(w) = (x_1, x_2, \dots, x_n)$.

Therefore, φ is surjective, concluding the

argument that $\mathbb{Z}(\{1, 2, \dots, n\}) \cong \mathbb{Z}^n$.

good.