MATH 332: HOMEWORK 6

Exercise 1. Construct your own *handcrafted, artisanal* semi-direct product: carefully select groups H and K from a small batch producer in the Cascades. Choose an explicit nontrivial homomorphism $\varphi : K \to \mathrm{Aut}(H)$. Write the product on $H \rtimes K$ explicitly. If you can, find an isomorphism from $H \rtimes K$ to a more familiar group. (Since your $H \rtimes K$ is *handcrafted* and *artisanal*, do not choose an example from the book or class.)

Problem 2. Let F be a field and let G denote the group of upper triangular matrices in $GL_n(F)$. Let U denote matrices in G with 1's along the diagonal, and let D denote diagonal matrices in G (arbitrary nonzero elements of F along the diagonal, 0's elsewhere). Prove that

$$G \cong U \rtimes D.$$

Problem 3. In the setup of Problem 2, let $n = 2$. In this case, $D \cong F^\times \times F^\times$ and $U \cong F$ (where F refers to the additive group of F). Explicitly determine the homomorphism

$$\varphi : F^\times \times F^\times \to \mathrm{Aut}(F)$$

which expresses G as the semi-direct product $U \rtimes D$.

Problem 4. Prove that $SL_2(\mathbb{F}_3)$ is isomorphic to the semi-direct product of Z_3 and Q_8. (It's up to you to determine if it's $Z_3 \rtimes Q_8$ or $Q_8 \rtimes Z_3$!)

Problem 5. Let F_1 and F_2 be free groups of the same finite free rank. Prove that $F_1 \cong F_2$. (*Bonus*: Do the same when F_1 and F_2 have free ranks of the same infinite cardinality.)

Problem 6. Find a presentation of A_4 with two generators (and prove that it is indeed a presentation of A_4).

Challenge 7. Define the *free abelian group* on the set S to be

$$\mathbb{Z}(S) = \langle S \mid [a, b] = 1 \text{ for all } a, b \in S \rangle.$$

Invent the universal property of $\mathbb{Z}(S)$ and prove that $\mathbb{Z}(S)$ satisfies it. Use the universal property to prove that $\mathbb{Z}(\{1, 2, \ldots, n\})$ is the n-fold direct product of $\mathbb{Z}$ with itself.

Date: 16.III.15.