MATH 332: HOMEWORK 5

Exercise 1. Exhibit all Sylow 2-subgroups and Sylow 3-subgroups of D_{12} and $S_3 \times S_3$.

Exercise 2. Exhibit two distinct Sylow 2-subgroups of S_5 and an element of S_5 that conjugates one into the other.

Exercise 3. Exhibit all Sylow 3-subgroups of $SL_2(\mathbb{F}_3)$.

Exercise 4. Prove that the subgroup of $SL_2(\mathbb{F}_3)$ generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the *unique* Sylow 2-subgroup of $SL_2(\mathbb{F}_3)$. (You may use your result from Problem 7 of Homework 2.)

Problem 5. Prove that a group of order 312 has a normal Sylow p-subgroup for some prime p dividing its order.

Problem 6. Let $P \in \mathrm{Syl}_p(G)$ and assume $N \trianglelefteqslant G$. Use the second Sylow theorem to prove that $P \cap N$ is a Sylow p-subgroup of N. Deduce that PN/N is a Sylow p-subgroup of G/N.

Problem 7. Prove that if N is a normal subgroup of G, then $n_p(G/N) \leqslant n_p(G)$.

Challenge 8. Prove that if there exists a chain of subgroups $G_1 \leqslant G_2 \leqslant G_3 \leqslant \cdots \leqslant G$ such that $G = \bigcup_{i=1}^{\infty} G_i$ and each G_i is simple, then G is simple.

Challenge 9. Find all normal subgroups of S_n for $n \geqslant 5$.

Challenge 10. Prove that A_n does not have a proper subgroup of index $< n$ for all $n \geqslant 5$.

Challenge 11 (Completely optional but possibly fun). Use a computer to find each $n < 10,000$ that is not a power of a prime and that has some prime divisor p such that n_p is not forced to be 1 for all groups of order n by the congruence condition of Sylow's theorem. For each such n, give the prime factorization of n and for each prime factor p list the permissible values of n_p (given the third part of Sylow's theorem). Say something interesting about the statistics of these numbers and potential implications for the distribution of simple groups.

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