

1) For  $\underline{3} = \{1, 2, 3\}$ , we have  $S_3 \subset \underline{3}^{\underline{3}}$  via  $\sigma(i, j) = (\sigma(i), \sigma(j))$

a) For an element  $(a, b) \in \underline{3}^{\underline{3}}$  and  $a \neq b$ , then any element  $(c, d) \in \underline{3}^{\underline{3}}$  such that  $c \neq d$  is in the orbit of  $(a, b)$ , via picking  $\sigma$  such that  $\sigma(a) = c$ . and  $\sigma(b) = d$ . Since  $c \neq d$ , such a mapping can be bijective, and thus such a bijection exists in  $S_3$ .

This orbit has a total orbit of  $3 \times 2 = 6$  elements from  $\underline{3}^{\underline{3}}$ .

The remaining orbit are the elements in  $\underline{3}^{\underline{3}}$  where  $a = b$ , where  $\sigma(a, b) = \sigma(a, a) = (\sigma(a), \sigma(a))$ . OK

b) permutation representation:  $e_{S_3} \rightarrow e_{\underline{3}^{\underline{3}}}$

$$(1 \ 2) \rightarrow ((1, 1)(2, 2))((1, 2)(2, 1))((1, 3)(2, 3))((3, 1)(3, 2))$$

$$(2 \ 3) \rightarrow ((2, 2)(3, 3))((2, 3)(3, 2))((1, 2)(1, 3))((3, 1)(3, 1))$$

$$(1 \ 3) \rightarrow ((1, 1)(3, 3))((1, 3)(3, 1))((1, 2)(3, 2))((2, 1)(2, 3))$$

$$(1 \ 2 \ 3) \rightarrow ((1, 1)(2, 2)(3, 3))((1, 2)(2, 3)(3, 1))((2, 1)(3, 2)(1, 3))$$

$$(3 \ 2 \ 1) \rightarrow ((1, 1)(3, 3)(2, 2))((1, 2)(3, 1)(2, 3))((2, 1)(1, 3)(3, 2))$$

c) First take  $a = (1, 1)$ . <sup>The</sup> stabilizer for  $a$  is  $\{e_{S_3}, (2 \ 3)\} \in S_3$ .

Now take  $a = (1, 2)$ . <sup>The</sup> stabilizer for  $a$  is  $e_{S_3} \in S_3$

Certainly the argument for there existing two orbits holds, as no specifications were placed on the present variables that were specific to  $\underline{3}^{\underline{3}}$ , except for the exponent of 2

Nice!

*Proof.* We know that  $\mathcal{O} = \{hxK \mid h \in H\}$ . If a given element  $h_1xk_1 \in HxK$ , then  $h_1xk_1 \in hxK$ , and we know that  $hxK \in \mathcal{O}$ , and if a given element  $h_1xk_1 \in hxK$  for some  $h$ , that element must be in  $HxK$ , so  $HxK = \bigcup_{gK \in \mathcal{O}} gK$ . OK

Let  $\mathcal{O} = \{Hxk \mid k \in K\}$  be the orbit of  $Hx$  under a right multiplication action by  $K$ . If a given element  $h_1xk_1 \in HxK$ , then  $h_1xk_1 \in Hxk$ , and we know that  $Hxk \in \mathcal{O}$ , and if a given element  $h_1xk_1 \in Hxk$  for some  $k$ , that element must be in  $HxK$ , so  $HxK = \bigcup_{Hg \in \mathcal{O}} Hg$ . ✓ Right.

Let  $h_1xk_1 \in HxK$  and  $h_2yk_2 \in HyK$ . If  $h_2yk_2 \in HxK$ , then  $hh_1xk_1k_2^{-1} = h_2yk_2$  for some arbitrary  $h \in H$ ,  $k \in K$ , since  $HxK$  is closed under left multiplication by elements of  $H$  and right multiplication by elements of  $K$ , since both of these are subgroups and therefore closed under multiplication. Therefore,  $y = h_2^{-1}hh_1xk_1kk_2^{-1}$  and so any arbitrary element  $h'yk' \in HyK$  is equal to  $h'h_2^{-1}hh_1xk_1kk_2^{-1}k'$ , and since  $h'h_2^{-1}hh_1 \in H$ ,  $k_1kk_2^{-1}k' \in K$ ,  $h'yk' \in HxK$ . So if  $HxK$  and  $HyK$  share a single element, they share all elements, so assuming  $HxK \neq HyK$ ,  $HxK \cap HyK = 0$ . And we know that for a given element  $g \in G$ , we can generate such a double coset  $HgK$  for any  $H, K \leq G$ , these sets partition  $G$ .

✓

3) Letting  $\mathbb{Q}_8 \times \mathbb{Q}_8$  by left multiplication, the permutation representation of the action is as follows:

Follows:  $i \rightarrow e$

$$-1 \rightarrow (1\ 5)(2\ 6)(3\ 7)(4\ 8)$$

$$i \rightarrow (1\ 2\ 5\ 6)(3\ 4\ 7\ 8)$$

$$j \rightarrow (1\ 3\ 5\ 7)(4\ 2\ 8\ 6)$$

$$K \rightarrow (1\ 4\ 5\ 8)(2\ 3\ 6\ 7).$$

Take  $\langle (1\ 2\ 5\ 6)(3\ 4\ 7\ 8), (1\ 3\ 5\ 7)(4\ 2\ 8\ 6) \rangle = P$

Let  $\varphi: \mathbb{Q}_8 \rightarrow P$ , via the mapping provided by the permutation representation  $\mathbb{Q}_8 \times \mathbb{Q}_8$ . To show

$\varphi$  is a homomorphism, note  $\varphi(i)^4 = ((1\ 2\ 5\ 6)(3\ 4\ 7\ 8))^4 = ((1\ 5)(2\ 6)(3\ 7)(4\ 8))^4$   
 $= e$ . Similarly, this holds for  $\varphi(j)^4 = ((1\ 3\ 5\ 7)(4\ 2\ 8\ 6))^4$ .

Furthermore,  $(\varphi(i)\varphi(j))^2 = (((1\ 2\ 5\ 6)(3\ 4\ 7\ 8))(1\ 3\ 5\ 7)(4\ 2\ 8\ 6))^2$   
 $= ((1\ 4\ 5\ 8)(2\ 3\ 6\ 7)) = \varphi(K^2) = (1\ 5)(2\ 6)(3\ 7)(4\ 8) = \varphi(K^2)$ .

Thus,  $\varphi(-1) = \varphi(i^2) = \varphi(i)^2 = \varphi(K^2) = \varphi(i; j)$ , which is sufficient to conclude  $\varphi$  is a homomorphism. Furthermore, since

$P = \langle \varphi(i), \varphi(j) \rangle$  and  $\varphi$  is a homomorphism,

$\varphi(i^a)\varphi(j^b)\varphi(i^c)\dots = \varphi(i^a j^b i^c \dots)$ . Thus,  $\varphi$  is surjective.

Since  $|i; j| = 8$ ,  $|\langle \varphi(i), \varphi(j) \rangle| = 8$ . Thus,

$P \cong \mathbb{Q}_8$ . Perfect.

Question 4a. The conjugation classes of  $D_8$ .

"Proof."

$$c_1 = \{1\}$$

$$c_2 = \{r^2\} \quad \checkmark$$

$$c_3 = \{s, r^2s\} \quad \checkmark$$

$$c_4 = \{rs, r^3s\} \quad \checkmark$$

$$c_5 = \{r, r^3\} \quad \checkmark$$

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Question 4b. The conjugation classes of  $Q_8$ .

*Proof.*

$$\begin{aligned}c_1 &= \{1\} \checkmark \\c_2 &= \{-1\} \checkmark \\c_3 &= \{i, -i\} \checkmark \\c_4 &= \{j, -j\} \checkmark \\c_5 &= \{k, -k\} \checkmark\end{aligned}$$

□

**Question 4c.** The conjugation classes of  $A_4$ .

*Proof.*

$$\begin{aligned}c_1 &= \{()\} \checkmark \\c_2 &= \{(12)(34), (13)(24), (14)(23)\} \checkmark \\c_3 &= \{(123), (132), (134), (143), (124), (142), (234), (243)\} \checkmark\end{aligned}$$

□

**Question 4d.** The conjugation classes of  $S_3 \times S_3$ .

*Proof.* First consider the conjugation classes of  $S_3$ . We proved in class that the conjugation classes refer to the decomposition of  $n = 3$ , so we get:

$$\begin{aligned}1 + 1 + 1 &\rightarrow c_1 = \{1\} \\1 + 2 &\rightarrow c_2 = \{(12), (13), (23)\} \\3 &\rightarrow c_3 = \{(123), (132)\}\end{aligned}$$

Since the permutations apply component wise, to find the conjugation classes of  $S_3 \times S_3$  we take the cartesian ~~cross~~ product:  $\{c_1, c_2, c_3\} \times \{c_1, c_2, c_3\}$ . ~~OK~~

□

**Question 5**

5. If  $G$  has exactly 2 conjugacy classes, at least one of them must be  $\{1\}$ . Then if there are precisely two, there must exist  $\kappa$ , the other conjugacy class.  $\kappa$  may be a singleton set or not. If it is a singleton set, then clearly  $G$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ . Suppose  $\kappa$  had two elements,  $g$  and  $h$ . Then  $hgh^{-1}$  is in  $\kappa$ . By assumption this element must either be  $g, h$  or 1, because if it weren't, then there would be more than one conjugacy class.  $hgh^{-1} = h \Rightarrow g = h$ , which cannot be the case,  $hgh^{-1} = 1$  implies that  $g = 1$ , which also cannot be the case and  $hgh^{-1} = g$  implies that  $hg = gh$ , which would mean that the group  $\{1, g, h\}$  is abelian, and that there should be conjugacy classes  $\{1\}, \{g\}, \{h\}$ , which also violates our assumptions. Therefore,  $\kappa = \{g\}$ , which means that there are precisely two elements to  $G$  and that  $G \cong \mathbb{Z}/2\mathbb{Z}$  ✓ nice

Problem 6:

$\sigma \in \text{Aut}(G)$ ;  $\varphi_g$  is conj. by  $g$ . Prove  $\sigma \varphi_g \sigma^{-1} = \varphi_{\sigma(g)}$ .

Let  $\bar{g} \in G$ .

$$\begin{aligned}
 \text{Then } (\sigma \varphi_g \sigma^{-1}) \bar{g} &= (\sigma \varphi_g)(\sigma(\bar{g})) \\
 &= \sigma \left( g \sigma^{-1}(\bar{g}) g^{-1} \right) \\
 &= \sigma(g) \cdot \sigma(\sigma^{-1}(\bar{g})) \cdot \sigma(g^{-1}) \\
 &= \sigma(g) \cdot \bar{g} \cdot \sigma(g^{-1}) \\
 &= \sigma(g) \cdot \bar{g} \cdot (\sigma(g))^{-1} \\
 &= \varphi_{\sigma(g)}(\bar{g}) \quad \checkmark
 \end{aligned}$$

Prove  $\text{Inn}(G) \trianglelefteq \text{Aut}(G)$ .

Since  $\sigma \varphi_g \sigma^{-1} = \varphi_{\sigma(g)}$  we have that

$$\sigma \{\varphi_g\}_{g \in G} \sigma^{-1} = \{\varphi_{\sigma(g)} \mid g \in G\}$$

$$\sigma \text{Inn}(G) \sigma^{-1} = \{\varphi_{\sigma(g)} \mid g \in G\} \quad \checkmark$$

Since  $\sigma$  is an automorphism,  $\sigma$  is bijective and  $\sigma(G) = G$ .

$$\text{So } \{\varphi_{\sigma(g)} \mid g \in G\} = \{\varphi_g\}_{g \in G} = \text{Inn}(G).$$

Hence, for all  $\sigma \in \text{Aut}(G)$ ,  $\sigma \text{Inn}(G) \sigma^{-1} = \text{Inn}(G)$ , so

$$\text{Inn}(G) \trianglelefteq \text{Aut}(G). \quad \checkmark$$

Sol.

7.  $n = 2$

$\langle x \rangle$	$\psi_1$
1	1
x	x



$n = 3$

$\langle x \rangle$	$\psi_1$	$\psi_2$
1	1	1
$x$	$x$	$x^2$
$x^2$	$x^2$	$x$

✓

$n = 4$

$\langle x \rangle$	$\psi_1$	$\psi_2$	$\psi_3$
1	1	1	1
$x$	$x$	$x^2$	$x^3$
$x^2$	$x^2$	1	$x^2$
$x^3$	$x^3$	$x^2$	$x$

✓

$n = 5$

$\langle x \rangle$	$\psi_1$	$\psi_2$	$\psi_3$	$\psi_4$
1	1	1	1	1
$x$	$x$	$x^2$	$x^3$	$x^4$
$x^2$	$x^2$	$x^4$	$x$	$x^3$
$x^3$	$x^3$	$x$	$x^4$	$x^2$
$x^4$	$x^4$	$x^3$	$x^2$	$x$

✓

$n = 6$

$\langle x \rangle$	$\psi_1$	$\psi_2$	$\psi_3$	$\psi_4$	$\psi_5$
1	1	1	1	1	1
$x$	$x$	$x^2$	$x^3$	$x^4$	$x^5$
$x^2$	$x^2$	$x^4$	1	$x^2$	$x^4$
$x^3$	$x^3$	1	$x^3$	1	$x^3$
$x^4$	$x^4$	$x^2$	$x^1$	$x^4$	$x^2$
$x^5$	$x^5$	$x^4$	$x^3$	$x^2$	$x$

✓

(5) Let  $X$  be a  $G$ -set and for some  $x \in X$  let  $G_x = \{g \cdot x \mid g \in G\}$  denote the orbit of  $x$  under  $G$ . Then  $G/G_x \cong_{G} Gx$  via  $gG_x \mapsto g \cdot x$ .

then a) If  $H \leq G$  and  $G \curvearrowright X$  via left multiplication, then a  $G$ -map  $F: G \rightarrow X$  maps to a  $G$ -map  $\bar{F}: G/H \rightarrow X$  given by  $\bar{F}(gH) = F(g)$  iff  $F(h) = F(1)$   $\forall h \in H$ .

~~MAP MAP~~ PROOF / (i) Let  $h \in Gx$  be s.t.  $h = g \cdot x = g' \cdot x$  for  $g, g' \in G$ ,  
 $\therefore g'^{-1}g \cdot x = g' \cdot x = x \Rightarrow g'^{-1}g \in G_x \Rightarrow g \cdot G_x = g' \cdot G_x$ , thus,  
 $: g \cdot G_x \mapsto g \cdot x$  is well-defined ✓ sure

- (i) (Surjective) Let  $\psi$  is surjective, since for any  $h \in G$  we have in some  $g' \in G$ , and  $\psi(g'G_x) = g'x = h \checkmark$
- (ii) (Injective) If  $g_x = g'x$ , then from part i) we know that  $gG_x = g'(G_x)$ , so suppose  $gG_x = g'G_x$ , then  $\exists h \in G_x$  s.t.  $g = g'h$ , then  $gx = g'hx = g'x$ , so  $\psi(gG_x) = \psi(g'(G_x)) \Leftrightarrow gG_x = g'G_x$  (i.e.  $\psi$  is injective) ~~✓~~ OK
- so  $\psi$  is injective & surjective, therefore  $\psi$  is bijective.  $\square$

9. Let  $G$  act on a set  $X$ , a set with a left  $G$ -action. Since the orbits of a group action constitute equivalence classes, we know that they partition  $X$ . Now assume  $X \in G\text{-set}$ . If  $X$  is a  $G$ -Set then the orbits of elements of  $X$  can be represented as  $x_1, x_2, \dots$  elements of each distinct partition. By the orbit stabilizer theorem  $G \setminus G_{x_i} \cong Gx_i$ . Clearly each  $Gx_i$  is distinct since orbits are partitions. Next, suppose  $hg \in G \setminus G_{x_i}$  and  $hg \in G \setminus G_{x_j}$ . Then  $bx_j = hg = ax_i$  where  $x_j \in G_{x_j}$  and  $x_i \in G_{x_i}$  and  $a, b \in G$ . Then  $(a^{-1}b)x_j = x_i \in G_{x_i}$ . Therefore,  $G_{x_i} = G_{x_j}$ , but since the  $x$ 's partition the set this implies that  $i = j$ . Therefore,  $\bigcup_{i=1,2,\dots} G \setminus G_{x_i}$  is a disjoint union. Define a map  $\varphi : \bigcup_{i=1,2,\dots} G \setminus G_{x_i} \rightarrow G$  such that  $\varphi$  figures out which portion of the disjoint union the element is in and then maps it via the isomorphism from that set to the corresponding  $Gx_i$ . Since each of these functions are isomorphic and none of them overlap,  $\varphi$  is a bijective homomorphism. Therefore,  $\bigcup_{i=1,2,\dots} G \setminus G_{x_i} \cong G$ . OK

11. We just need to make sure that  $Gx \cong_G G/G_x \Rightarrow Gx \cong G/G_x$ . We know that  $f$  is bijective, so we just need to make sure that it's a homomorphism. For  $g, h \in G_x$ ,

$$f(gG_xhG_x) = f(ghG_x) = gh \cdot x = g \cdot h \cdot x = f(gG_x) \cdot f(hG_x)$$

Thus there is an isomorphism, so we know that  $|Gx| = |G/G_x| = \frac{|G|}{|G_x|}$  via Lagrange's theorem.