MATH 332: HOMEWORK 4

Problem 1. Let $\underline{3} = \{1,2,3\}$ and let S_3 act on $\underline{3}^2 = \underline{3} \times \underline{3}$ via $\sigma \cdot (i,j) = (\sigma(i), \sigma(j))$.

(a) Find the orbits of S_3 on $\underline{3}^2$.

(b) For each $\sigma \in S_3$ find the cocycle decomposition of σ under this action. (*I.e.*, the action affords a permutation representation $S_3 \to S_{\underline{3}^2} \cong S_9$ where the final isomorphism is given by choosing a labelling by $\underline{9}$ of the elements of $\underline{3}^2$. Find the cycle decomposition of the image of each elements of S_3 in S_9.)

(c) For each orbit $\mathcal{O} \subset \underline{3}^2$ of $S_3 \circlearrowright \underline{3}^2$, pick some $a \in \mathcal{O}$ and find the stabilizer of a in S_3.

Bonus: Can you generalize any of this to $S_n \circlearrowright \underline{n}^2$?

Problem 2 (Double your cosets, double your fun). Let $H, K \leqslant G$ be subgroups of a group G. For each $x \in G$ define the HK *double coset* of x in G to be

$$HxK = \{hxk \mid h \in H, k \in K\}.$$

(a) Let H act by left multiplication on the set of left cosets of K (*i.e.* $H \circlearrowright G/K$ via left multiplication) and let $\mathcal{O}$ be the orbit of xK under this action. Prove that

$$HxK = \bigcup_{gK \in \mathcal{O}} gK.$$

(b) Prove that HxK can also be written as a union of right cosets of H.

(c) Show that the set of HK double cosets partitions G.

(d) Prove that

$$|HxK| = |K| \cdot [H : H \cap xKx^{-1}] = |H| \cdot [K : K \cap x^{-1}Hx].$$

Problem 3. Use the left regular representation of Q_8 (*i.e.* the permutation representation of $Q_8 \circlearrowright Q_8$ via left multiplication) to produce two elements of S_8 which generate a subgroup of S_8 isomorphic to the quaternion group Q_8.

Problem 4. Find all conjugacy classes and their sizes in the following groups:

(a) D_8

(b) Q_8

(c) A_4

(d) $S_3 \times S_3$.

Problem 5. Find all finite groups which have exactly two conjugacy classes. *Bonus*: Exactly three conjugacy classes? *Hint*: The class equation.

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Problem 6. Let G be a group. If $\sigma \in \operatorname{Aut}(G)$ and φ_g is conjugation by g, prove that $\sigma\varphi_g\sigma^{-1} = \varphi_{\sigma(g)}$. Deduce that $\operatorname{Inn}(G) \trianglelefteq \operatorname{Aut}(G)$. (The group $\operatorname{Aut}(G)/\operatorname{Inn}(G)$ is called the *outer automorphism group* of G and is denoted $\operatorname{Out}(G)$.)

Problem 7. Let $G = \langle x \rangle$ be a cyclic group of order n. Recall that $\operatorname{Aut}(G) \cong (\mathbb{Z}/n\mathbb{Z})^\times$ via the assignment $a \in (\mathbb{Z}/n\mathbb{Z})^\times \mapsto \psi_a$, where $\psi_a(x) = x^a$. For $n = 2, 3, 4, 5, 6$, write out explicitly what ψ_a does to the elements $1, x, x^2, \ldots, x^{n-1}$ of G.

Fix a group G. Let $G\text{-}\mathrm{Set}$ denote the collection of *left G-sets*, that is, sets X equipped with a left G-action $G \circlearrowright X$. A *G-equivariant map* of G-sets (or just *G-map* for short) is a function $f : X \to Y$ between G-sets such that $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$, $x \in X$. A function between G-sets is a *G-isomorphism* if it is a bijective G-map. Two G-sets X, Y are *G-isomorphic* if there exists a G-isomorphism $X \to Y$; in this case, we write $X \cong_G Y$. It is easy to check that $\cong_G$ is an equivalence relation on $G\text{-}\mathrm{Set}$ (do so!).

The following two problems give us a way to think about G-isomorphism classes in $G\text{-}\mathrm{Set}$. By the end of Problem 9 we will see that every G-set X can be written (up to G-isomorphism) as a disjoint union

$$X \cong_G \coprod_{i \in I} G/H_i$$

where $\{H_i \mid i \in I\}$ is a collection (possibly with redundancy) of subgroups $H_i \leqslant G$. (Of course, $G \circlearrowright G/H_i$ via $g \cdot xH_i = (gx)H_i$ for $g, x \in G$.) If you complete the optional Problem 10, you will find out which coset G-sets G/H are G-isomorphic to each other, thus completely settling the problem of G-isomorphism classes in $G\text{-}\mathrm{Set}$ (each G-isomorphism class is determined by a list [with multiplicity] of conjugacy classes of subgroups of G up to permutation). In fact, if you return to Problem 10 after you've learned what a *category* is, you will discover that you now know the structure of the category of G-sets.

Problem 8. Prove the *orbit-stabilizer theorem*: Let X be a G-set and for $x \in X$ let $Gx = \{g \cdot x \mid g \in G\}$ denote the orbit of x under G. Then $G/G_x \cong_G Gx$ via $gG_x \mapsto g \cdot x$. You are welcome to proceed via the following outline:

(a) Show that if $H \leqslant G$ and $G \circlearrowright G$ via left multiplication, then a G-map $F : G \to X$ extends to a G-map $\bar{F} : G/H \to X$ given by $\bar{F}(gH) = F(g)$ if and only if $F(h) = F(1)$ for all $h \in H$.
(b) Use (a) to show that $f : G/G_x \to Gx$ given by $f(gG_x) = g \cdot x$ is well-defined.
(c) Show that f is bijective. (Surjective should be easy; injective requires a slightly more substantial argument.)

Proof. (a) First suppose that $F(h) = F(1)$ for all $h \in H$. We must show that $F(g) = F(g')$ whenever $g, g' \in gH$. Since $g' \in gH$, $g' = gh$ for some

$h \in H$. Thus $F(g') = F(gh) = gF(h) = gF(1) = F(g)$, as desired. (Here we have used the fact that $F(g_1 g_2) = g_1 F(g_2)$ for all $g_1, g_2 \in G$ twice.)

Now suppose that $\bar{F}(gH) = F(g)$ is well-defined. Then $F(g) = F(gh)$ for all $g \in G$, $h \in H$. In particular, if $g = 1$, we get $F(1) = F(h)$ for all $h \in H$, as desired.

Parts (b) and (c) are still up to you!

□

Problem 9. Let $G \circlearrowright X$ be a set X with a left G-action. Show that the orbits of elements of X partition X. Now assume $X \in G\text{-}\mathrm{Set}$ and use the orbit-stabilizer theorem of Problem 8 to show that X is G-isomorphic to a disjoint union of G-sets of the form G/H, $H \leqslant G$. (Here G/H has the obvious left G-action.)

Problem 10 (*Bonus* – the category of G-orbits). Let $H, K \leqslant G$ be subgroups of a group G. Prove the following statements:

(a) There exists a G-map $G/H \to G/K$ if and only if H is subconjugate to K. (Here *subconjugate* means that H is conjugate to a subgroup of K, *i.e.*, there exists $x \in G$ such that $x^{-1}Hx \leqslant K$.)
(b) Every G-map $G/H \to G/K$ has the form $R_x : gH \mapsto gxK$ where $x \in G$ such that $x^{-1}Hx \leqslant K$.
(c) The maps $R_x = R_y$ if and only if $x^{-1}y \in K$.
(d) The G-sets G/H and G/K are G-isomorphic if and only if H and K are conjugate in G.

We conclude with a cute and useful application of Problem 8.

Problem 11. Suppose G is a finite group and X is a finite G-set. Use the orbit-stabilizer theorem (Problem 8) and Lagrange's theorem to prove that for all $x \in X$,

$$|Gx| = \frac{|G|}{|G_x|}.$$