MATH 332: HOMEWORK 3

Problem 1. Prove that if H and K are finite subgroups of G whose orders are relatively prime, then $H \cap K = 1$.

Problem 2. Use Lagrange's Theorem in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ to prove *Fermat's little theorem*: if p is prime, then $a^p \equiv a \pmod{p}$ for all $a \in \mathbb{Z}$.

Problem 3. Prove that if N is a normal subgroup of the finite group G and $(|N|, [G : N]) = 1$, then N is the unique subgroup of G of order $|N|$.

Problem 4. Prove that if H is a normal subgroup of G of prime index p, then for all $K \leq G$, either

(i) $K \leq H$ or
(ii) $G = HK$ and $[K : K \cap H] = p$.

Problem 5. Let p be a prime and let $\mu_{p^\infty}(\mathbb{C})$ be the group of p-power roots of unity in $\mathbb{C}$. Show that the map $z \mapsto z^p$ is a surjective homomorphism. Deduce that $\mu_{p^\infty}(\mathbb{C})$ is isomorphic to a proper quotient of itself. (This means that $\mu_{p^\infty}(\mathbb{C}) \cong \mu_{p^\infty}(\mathbb{C})/N$ for some $1 \neq N \trianglelefteq \mu_{p^\infty}(\mathbb{C})$.)

Problem 6. Suppose that N is a normal subgroup of G, let $i : N \to G$ denote the inclusion of N into G, let $\pi : G \to G/N$ denote the natural projection, and $\varphi : G \to H$ be a homomorphism to a group H. Consider the diagram

$$\begin{tikzcd}
N \arrow[r, "i"] \arrow[dr] & G \arrow[r, "\pi"] \arrow[d, "\varphi"] & G/N \arrow[dl, dashed, "\bar{\varphi}"] \\
& H &
\end{tikzcd}$$

Prove that a homomorphism $\bar{\varphi} : G/N \to H$ making the diagram commute exists if and only if $\varphi \circ i$ is the trivial homomorphism. (Note that the condition $\varphi \circ i = 1$ is equivalent to $\varphi(N) = 1$, which is in turn equivalent to $\ker \varphi \leq N$.) Show additionally that when such a $\bar{\varphi}$ exists, it is unique. *Bonus*: Show that this property of $\pi : G \to G/N$ uniquely characterizes it. In other words, show that if $p : G \to K$ is any other group homomorphism satisfying the same "unique extension" property, then there is a unique isomorphism $\Phi : G/N \to K$ such that $\Phi \circ \pi = p$. (This justifies the moniker under which the above property goes: the *universal property* of the quotient map $G \to G/N$.)

Problem 7. Let M and N be normal subgroups of G such that $G = MN$. Prove that $G/(M \cap N) \cong (G/M) \times (G/N)$. (You may want to use your result from Problem 6.)

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Problem 8. Prove that subgroups and quotient groups of a solvable group are solvable.

Problem 9. Prove that σ^2 is an even permutation for any permutation σ.

Problem 10. Show that $S_n = \langle (1\ \ 2), (1\ \ 2\ \ 3\ \ \cdots\ \ n) \rangle$ for all $n \geq 2$.

Problem 11. Prove that the group of rigid motions of the tetrahedron is isomorphic to A_4.