MATH 332: HOMEWORK 2

Problem 1. Show that the following rules constitute (left) group actions on the specified sets:

(a) Let F be a field and $F^\times = F \smallsetminus \{0\}$ the multiplicative group of nonzero elements of F. Then $F^\times$ acts on F via $g \cdot a = ga$ where $g \in F^\times$, $a \in F$, and ga is the usual product in F of the two field elements g and a.

(b) The additive group $\mathbb{R}$ acts on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ via $r \cdot (x, y) = (x + ry, y)$.

(c) The group $GL_2(\mathbb{R})$ of 2×2 invertible matrices with real entries acts on $\mathbb{R}^2$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

(Here we have written $(x, y) \in \mathbb{R}^2$ as a column vector.)

Bonus: What is the relationship between the action in (b) and the action in (c)?

Problem 2. Prove that the kernel of an action of the group G on a set A is the same as the kernel of the corresponding permutation representation $G \to S_A$.

Problem 3. Assume n is an even positive integer and show that D_{2n} acts on the set consisting of pairs of opposite vertices of a regular n-gon. Find the kernel of this action.

Problem 4. Let G be an abelian group. Prove that $\{g \in G \mid |g| < \infty\}$ is a subgroup of G (called the *torsion subgroup* of G). Give an explicit example where this set is not a subgroup when G is non-abelian.

Problem 5. Fix some $n \in \mathbb{Z}$ with $n > 1$. Find the torsion subgroup of $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$. Show that the set of elements of infinite order together with the identity is *not* a subgroup of this direct product.

Problem 6. Prove that if H and K are subgroups of G, then so is their intersection $H \cap K$. *Bonus*: Prove that the intersection of an arbitrary nonempty collection of subgroups of G is again a subgroup of G. (Do not assume that the collection of subgroups is countable.)

Problem 7. Prove that the subgroup of $GL_2(\mathbb{F}_3)$ is the subgroup generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is isomorphic to the quaternion group Q_8.

Problem 8. A group H is called *finitely generated* if there is a finite set A such that $H = \langle A \rangle$.

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(a) Prove that every finite group is finitely generated.
(b) Prove that $\mathbb{Z}$ is finitely generated.
(c) Prove that every finitely generated subgroup of the additive group $\mathbb{Q}$ is cyclic. [If H is a finitely generated subgroup of $\mathbb{Q}$, show that $H \leq \langle 1/k \rangle$ where k is the product of all the denominators which appear in a set of generators for H.]
(d) Prove that $\mathbb{Q}$ is not finitely generated.

Problem 9. Let $\varphi : G \to H$ be a homomorphism and let E be a subgroup of H. Prove that $\varphi^{-1}(E) \leq G$. If $E \trianglelefteq H$, prove that $\varphi^{-1}(E) \trianglelefteq G$. Deduce that $\ker \varphi \trianglelefteq G$.

Problem 10. Define $\varphi : \mathbb{C}^\times \to \mathbb{R}^\times$ by $\varphi(a + bi) = a^2 + b^2$. Prove that φ is a homomorphism and find its image. Describe the kernel and fibers of φ geometrically (as subsets of the plane).

Problem 11. Consider the additive quotient group $\mathbb{Q}/\mathbb{Z}$.

(a) Show that every coset of $\mathbb{Z}$ in $\mathbb{Q}$ contains exactly one representative $q \in \mathbb{Q}$ in the range $0 \leq q < 1$.
(b) Show that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order; nevertheless, there are elements of arbitrarily large order.
(c) Show that $\mathbb{Q}/\mathbb{Z}$ is the torsion subgroup of $\mathbb{R}/\mathbb{Z}$.
(d) Prove that $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the multiplicative group of roots of unity in $\mathbb{C}^\times$.

Problem 12. Let $SL_n(F) = \{A \in GL_n(F) \mid \det A = 1\}$. Prove that $SL_n(F) \trianglelefteq GL_n(F)$ and describe the isomorphism type of the quotient group

$$GL_n(F)/SL_n(F).$$

(You may assume that $\det : GL_n(F) \to F^\times$ is a homomorphism.)