MATH 332: HOMEWORK 11

Exercise 1. Compute

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/65\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/91\mathbb{Z}, \mathbb{Z}/131\mathbb{Z})$$

as an abelian group.

Exercise 2. Find, with proof, the number of finitely generated abelian groups of order 100. Do the same for finitely generated abelian groups of order 576.

Problem 3. Suppose R is a commutative ring and that M and N are free R-modules of ranks m and n, respectively. Show that $M \otimes_R N$ is a free R-module of rank mn. Find a basis for $M \otimes_R N$ in terms of bases $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ of M and N, respectively.

Problem 4. Let M be a module over an integral domain R.

(a) Suppose x is a nonzero torsion element in M. Show that x and 0 are linearly dependent. Conclude that the rank of $\mathrm{Tor}(M)$ is 0, so in particular, any torsion R-module has rank 0.
(b) Show that the rank of M is the same as the rank of $M/\mathrm{Tor}(M)$.
(c) If R is a PID and M is a finitely generated R-module, describe the structure of $M/\mathrm{Tor}(M)$.

Problem 5. Let R be a PID and let M be a finitely generated R-module. By the structure theorem,

$$M \cong R^r \oplus R/(a_1) \oplus \cdots \oplus R/(a_d)$$

for some $r, d \geqslant 0$ and a_i nonzero, nonunit elements of R such that $a_1 \mid a_2 \mid \cdots \mid a_d$. As such, there is a canonical map $R^{r+d} \to M$. Describe this map and its kernel; in particular, prove that the kernel is a free R-module.

Problem 6. Let G be the quotient group $\mathbb{Q}/\mathbb{Z}$. Is G a free $\mathbb{Z}$-module? Torsion-free? Finitely-generated?

Challenge 7. Let A be a subgroup of $\mathbb{R}^n$ such that in each ball in $\mathbb{R}^n$ there are only finitely many elements of A. Show that A is a free abelian group on at most n generators.

Challenge 8. Find a commutative ring R and finitely generated R-module M such that there is no R-module homomorphism from a finite rank free R-module to M with free kernel. (Compare with the situation in Problem 5.)

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