

1) Let $m \in \mathbb{N}$. We know $0 = 0 + 0$ and $0 + 0n = 0m = \underbrace{(0+0)}_n = 0m + 0m$.
And $0 + 0m = 0n + 0m \Rightarrow 0 = 0m$. ✓

Axiom: $(r+s)m = rm + sm$

We have $0 = 1 + (-1)$, so $0n = (1 + (-1))n = 1n + (-1)n$. We have $0 = 0n$, so
we get $0 = 1n + (-1)n$ and $0 = n + (-1)n$ axiomatically. We add n to both
sides to get $-n + 0 = -n + n + (-1)n \Rightarrow -n = (-1)n$. // ✓

2. The $F[x]$ subspaces we specify are precisely the T -stable subspaces of V .
By the problem statement $T(x_1, x_2) = (0, x_2)$

(S)

Choose some $(x_1, x_2) \in \mathbb{R}^2$. $T(x_1, x_2) = (0, x_2) \in \mathbb{R}^2$

Thus $T(\mathbb{R}^2) \subseteq \mathbb{R}^2$, so \mathbb{R}^2 is a T -stable subspace.

$O = \{(0,0)\}$. $T(0,0) = (0,0) \in O$. So $T(O) = O$, therefore O is a T -stable subspace.

$0 \times \mathbb{R} = \{(0, x_2) \mid x_2 \in \mathbb{R}\}$. For all $(0, x_2) \in 0 \times \mathbb{R}$, $T(0, x_2) = (0, x_2) \in 0 \times \mathbb{R}$.

Thus $T(0 \times \mathbb{R}) = 0 \times \mathbb{R}$, which means that $0 \times \mathbb{R}$ is also a T -stable subspace.

$\mathbb{R} \times O = \{(x_1, 0) \mid x_1 \in \mathbb{R}\}$. For all $(x_1, 0) \in \mathbb{R} \times O$, $T(x_1, 0) = (0, 0) \in \mathbb{R} \times O$.

Thus $T(\mathbb{R} \times O) = \{(0,0)\} \subseteq \mathbb{R} \times O$. So $\mathbb{R} \times O$ is T -stable.

Note that these are the only subsets of V which are closed under multiplication by any $t \in \mathbb{R}$. If $(a, b) \in V$, $t(a, b) = (ta, tb)$. If $a \neq 0$, then we can make this anything, by scaling it by acting $r = \frac{tb}{a}$. Similar for the second coordinate. Thus the only possible subsets of V which are closed under multiplication are O , $0 \times \mathbb{R}$, $\mathbb{R} \times O$, and V . Thus, we need not consider any other subsets.

 Problem 3. If N is a submodule of an R -module M , the *annihilator* of N in R is defined to be

$$\text{Ann}(N) = \{r \in R \mid rn = 0 \text{ for all } n \in N\}.$$

Prove that $\text{Ann}(N)$ is a 2-sided ideal of R .

If $n \in N$, then $rn \in N \forall n \in N, r \in R$ by the second part of the submodule criterion. So if $rn = 0 \forall n \in N$, then for such r and arbitrary $r' \in R$, we have $rr'n = rn' = 0$ and $r'rn = r'0 = 0$. So for all $r \in \text{Ann}(N), r' \in R$, $rr' \in \text{Ann}(N)$ and $r'r \in \text{Ann}(N)$, so $\text{Ann}(N)$ is closed under multiplication by arbitrary elements of the ring. Now if $r, r' \in \text{Ann}(N)$, we have $(r+r')n = rn + r'n = 0 + 0 = 0$ for all $n \in N$, so $\text{Ann}(N)$ is closed under addition, and so it is a (2-sided) ideal of R ✓

 Problem 4. Let A be any \mathbb{Z} -module, let a be any element of A , and let n be any positive integer.

- (a) Prove that $\varphi_a : \mathbb{Z}/n\mathbb{Z} \rightarrow A$ given by $\varphi(\bar{k}) = ka$ is a well-defined \mathbb{Z} -module homomorphism if and only if $na = 0$.
- (b) Prove that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong {}_nA$, where ${}_nA = \{a \in A \mid na = 0\}$.

Answer:

- (a) Suppose first that ϕ is well defined \mathbb{Z} module homomorphism. Then $\phi(b + nk) = \phi(b + nl)$ for any $k, l \in \mathbb{Z}$. Since ϕ is a homomorphism $\phi(b) + \phi(n)\phi(l) = \phi(b) + \phi(n)\phi(l)$. This can only be the case if $\phi(n)\phi(l) = \phi(n)\phi(k)$. As this must hold even if $l \neq k$, $\phi(n) = na = 0$. Now suppose $na = 0$. Note first that $\phi((b + nk) + \mathbb{Z}/n\mathbb{Z}) = a(b + nk) + = ab + ank = ab = \phi(b + \mathbb{Z}/n\mathbb{Z})$. Thus, the mapping is same given different coset representations on \mathbb{Z} , meaning the mapping is well-defined. Furthermore, $\phi(b + nk) + \phi(b + nl) = ba + ba = (2b)a = \phi(b + b + n(k + l))$, implying addition of elements in $\mathbb{Z}/n\mathbb{Z}$ is preserved under this mapping. Furthermore, for any $x \in \mathbb{Z}$, $\phi(x(b + n\mathbb{Z})) = xbk = x\phi(b + n\mathbb{Z})$, implying scalar multiplication is preserved, concluding the proof that ϕ is a well defined homomorphism if $na = 0$. \checkmark
- (b) Since $\mathbb{Z}/n\mathbb{Z}$ is a cyclic group, given a homomorphism $\phi \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A)$, ϕ is completely determined by the value of $\phi(1 + n\mathbb{Z})$. Therefore, all homomorphism in $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A)$ are of the form $\phi(k(1 + n\mathbb{Z})) = ka$, or the form described above. Thus, define a map $f : \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \rightarrow {}_nA$, by $(\phi(k + n\mathbb{Z})) \mapsto ak$, where $a \in {}_nA$. Note first that this mapping is well-defined over all of the domain of f by the previous argument. Secondly, note that given $a, b \in {}_nA$, $(\phi_a + \phi_b)(k + n\mathbb{Z}) = \phi_a(k + n\mathbb{Z}) + \phi_b(k + n\mathbb{Z}) = ak + bk = (a + b)k$. Thus, $f(\phi_a + \phi_b) = f(\phi_{a+b}) = a + b = f(\phi_a) + f(\phi_b)$. Furthermore, given $x \in \mathbb{Z}$, $x\phi_a(k + n\mathbb{Z}) = x(ka)$. Thus, $f(x\phi_a) = xa = xf(\phi_a)$, concluding the proof that the mapping is a

good

module homomorphism. This mapping is surjective since for all $a \in {}_nA$, ϕ_a is a homomorphism in $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A)$, meaning that $f(\phi_a) = a$. Now let $f(\phi_a) = f(\phi_b)$. Since $f(\phi_a) = a$ and $f(\phi_b) = b$, $a = b$. Thus, $\phi_b = \phi_a$, concluding the proof that f is injective and thus an isomorphism between \mathbb{Z} modules, meaning $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong {}_nA$.

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By 4(b) we have ~~Hom_Z(Z/mZ, Z/mZ)~~ $\cong {}_n\text{Ann}(Z/mZ)$. ~~For some reason~~

~~never mind~~, let $d = \gcd(m, n)$ and write $m = ad$, $n = bd$ for some
~~prime integers~~ prime integers a and b . Now, $m \cdot \bar{b} = (ad) \bar{b} = a(\bar{d} \cdot \bar{b}) \cong {}_{a \cdot d}^{\text{mod}} \bar{a} \cdot \bar{b} = \bar{0} \Rightarrow \bar{b} \in {}_n\text{Ann}(Z/mZ)$.

$\varphi: Z \rightarrow {}_n\text{Ann}(Z/mZ)$ by $\varphi(1) = \bar{b}$, so we know that φ is a \mathbb{Z} -module homomorphism. Since $\bar{b} \in {}_n\text{Ann}(Z/mZ)$, φ is ~~surjective~~ surjective (every element in the annihilator must be an multiple of \bar{b}).

$\ker(\psi) = \{k \mid \psi(k) = 0\} = \{k \in \mathbb{Z}/m\mathbb{Z} \mid d \mid k\} = \{k \mid (d, k) = 1\} = \mathbb{Z}/(d)\mathbb{Z}$, so by

the first isomorphism, $\text{Ann}(\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(d)\mathbb{Z} \cong \mathbb{Z}/(n, m)\mathbb{Z}$, so by 16,

$\mathbb{Z}/(n, m)\mathbb{Z} \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}; \mathbb{Z}/m\mathbb{Z})$. \square good.



Challenge 6. Assume R is commutative. Prove that $R^n \cong R^m$ if and only if $n = m$, i.e., two free R -modules of the same rank are isomorphic if and only if they have the same rank. (See Exercise 2 on p.356 for a hint.)

Recall that $R^n/IR^n = (R \times R \times \cdots \times R)/(I \times I \times \cdots \times I) = (R/I) \times \cdots \times (R/I)$ (n times). For I a maximal ideal of R , this means we have a field $F \times F \times \cdots \times F$ (n times). Now for $R^n \cong R^m$, we must have the same relation $R^n/IR^n \cong R^m/IR^m = F \times F \times \cdots \times F$ (m times), and for these two vector spaces F^n and F^m to be isomorphic they must have the same dimension: $F^n \cong F^m \iff n = m$. ✓

7) Let $MN : N$ be finitely generated. So $\exists X \subseteq N$ s.t. $N = RX$. Similarly,
 $\exists Y \subseteq M$ s.t. $MN = R\{y_1, y_2, \dots, y_n\}$. Let $X = \{x_1, x_2, \dots, x_n\}$, $Y = \{y_1, y_2, \dots, y_n\}$.

Then $N = \{r_1, r_2, \dots, r_n | r_i \in R\} : MN = \{s_1, s_2, \dots, s_n | s_i \in R\}$.
Let $m \in M$, then $m + n \in MN$, so $\exists s_1, \dots, s_n$ s.t. $m + n = s_1y_1 + \dots + s_ny_n$.
By our definition of N , $\exists r_1, \dots, r_n \in R$ s.t. $m = s_1y_1 + \dots + s_ny_n + r_1x_1 + \dots + r_nx_n \in R(X \cup Y)$.
must be finite, so M is generated by a finite set **good**

8. Note that $(a+bi) \otimes (c+di) = (a+bi) \otimes c + (a+bi) \otimes di$. Since this is a tensor over \mathbb{R} , we can't move the i from one side of the tensor product to another. Thus all elements of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ can be expressed as Right.

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$$z \otimes 1 + y \otimes i, \quad y, z \in \mathbb{C}.$$

This $\otimes \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ as a \mathbb{C} -vector space has dimension 2 because it is precisely $\mathbb{C}\{1 \otimes 1, 1 \otimes i\}$.

$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ has a clear addition operation. $(a \otimes 1 + b \otimes i) + (c \otimes 1 + d \otimes i) = (a+c) \otimes 1 + (b+d) \otimes i$

Since multiplication in \mathbb{R} is commutative, this is clearly abelian in addition as well.

We can define multiplication similarly to the way it's done in \mathbb{C} , so that $(a \otimes 1 + b \otimes i) \cdot (c \otimes 1 + d \otimes i) = (ac - bd) \otimes 1 + (ad + bc) \otimes i$.

This is commutative. It is also distributive.

~~$$(a \otimes 1 + b \otimes i) \cdot (c \otimes 1 + d \otimes i + e \otimes 1 + f \otimes i)$$~~

$$= [a(c+e) - b(d+f)] \otimes 1 + [a(d+f) + b(c+e)] \otimes i = (a \otimes 1 + b \otimes i) \cdot (c \otimes 1 + d \otimes i) + (a \otimes 1 + b \otimes i) \cdot (e \otimes 1 + f \otimes i).$$

This is actually basically \mathbb{C} . ✓