## MATH 332: HOMEWORK 10

Throughout, $R$ is a (not necessarily commutative) ring with $1$ unless otherwise specified.

*Exercise* 1. Suppose $M$ is a left $R$-module. Prove that $0m = 0$ and $-1m = -m$ for all $m \in M$.

*Exercise* 2. Let $F = \mathbb{R}$, let $V = \mathbb{R}^2$, and let $T$ be the linear transformation $V \to V$ which is projection onto the second coordinate. Show that $V$, $0$, $\mathbb{R} \times 0$, and $0 \times \mathbb{R}$ are the only $F[x]$-submodules of $V$ for this choice of $T$.

*Problem* 3. If $N$ is a submodule of an $R$-module $M$, the *annihilator* of $N$ in $R$ is defined to be

$$\mathrm{Ann}(N) = \{r \in R \mid rn = 0 \text{ for all } n \in N\}.$$

Prove that $\mathrm{Ann}(N)$ is a 2-sided ideal of $R$.

*Problem* 4. Let $A$ be any $\mathbb{Z}$-module, let $a$ be any element of $A$, and let $n$ be any positive integer.

(a) Prove that $\varphi_a : \mathbb{Z}/n\mathbb{Z} \to A$ given by $\varphi(\bar{k}) = ka$ is a well-defined $\mathbb{Z}$-module homomorphism if and only if $na = 0$.

(b) Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong {}_nA$, where ${}_nA = \{a \in A \mid na = 0\}$.

*Problem* 5. Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n,m)\mathbb{Z}$.

*Challenge* 6. Assume $R$ is commutative. Prove that $R^n \cong R^m$ if and only if $n = m$, *i.e.*, two free $R$-modules of the same rank are isomorphic if and only if they have the same rank. (See Exercise 2 on p.356 for a hint.)

*Problem* 7. Let $N$ be a submodule of $M$. Prove that if both $M/N$ and $N$ are finitely generated, then so is $M$.

*Problem* 8. Determine the dimension of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ as a $\mathbb{C}$-vector space. Determine the natural ring structure on $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$.

*Date*: 20.IV.15.