MATH 332: HOMEWORK 1

Unless otherwise specified, $(G, \cdot)$ is a group. We will often refer to $(G, \cdot)$ as simply G, and write ab for $a \cdot b$ when $a, b \in G$.

Problem 1. For a positive integer n, let $\mu_n(\mathbb{C})$ denote the set of complex numbers whose n-th power is 1. Prove that $(\mu_n(\mathbb{C}), \cdot)$ is a group where $\cdot$ is the usual multiplication of complex numbers. (This group is frequently called the *group of n-th roots of unity*.) Also prove that $(\mu_n(\mathbb{C}), +)$ is not a group where $+$ is the usual addition of complex numbers.

Problem 2. For a positive integer n, let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ and define an operation $\boxplus : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n$ such that $a \boxplus b$ is the remainder of $a + b$ after dividing by n. Show that $\boxplus$ is well-defined and that $(\mathbb{Z}_n, \boxplus)$ is a group. (Later, we will write $(\mathbb{Z}/n\mathbb{Z}, +)$ for $(\mathbb{Z}_n, \boxplus)$ and call it the *group of integers mod n*.) Also prove that $(\mathbb{Z}_n, \boxtimes)$ is not a group where $a \boxtimes b$ is the remainder of $a \cdot b$ after dividing by n.

Problem 3. Prove that $(\mu_n(\mathbb{C}), \cdot)$ is isomorphic to $(\mathbb{Z}_n, \boxplus)$. *Bonus*: Invent other groups which are isomorphic to $\mu_n(\mathbb{C})$ and $\mathbb{Z}_n$.

Problem 4. Let x be an element of G. Suppose $|x| = n$ for some positive integer n. Prove that $x^{-1} = x^{n-1}$.

Problem 5. For $x, y \in G$, prove that $xy = yx$ if and only if $y^{-1}xy = x$ if and only if $x^{-1}y^{-1}xy = 1$.

Problem 6. Compute the order of each of the elements of each of the following groups: D_6, D_8, and D_{10}.

Problem 7. Let T be the group of rigid motions (*aka* rotations) of a regular tetrahedron. Show that $|T| = 12$. *Bonus*: Let C, O, I, and D be the groups of rigid motions of the regular cube, octahedron, icosahedron, and dodecahedron. Find $|C|$, $|O|$, $|I|$, and $|D|$.

Problem 8. Suppose $\sigma, \tau \in S_{15}$ have cycle decompositions

$$\sigma = (1\ \ 13\ \ 5\ \ 10)(3\ \ 15\ \ 8)(4\ \ 14\ \ 11\ \ 7\ \ 12\ \ 9)$$
$$\tau = (1\ \ 14)(2\ \ 9\ \ 15\ \ 13\ \ 4)(3\ \ 10)(5\ \ 12\ \ 7)(8\ \ 11).$$

Find the cycle decompositions of σ^2, $\sigma\tau$, $\tau\sigma$, and $\tau^2\sigma$.

Problem 9. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers. Prove that $S_{\mathbb{N}}$ is an infinite group. *Bonus*: Can you say anything more precise about the cardinality of $S_{\mathbb{N}}$?

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Problem 10. If A, B, and C are groups, prove that

$$A \times B \cong B \times A \quad \text{and} \quad A \times (B \times C) \cong (A \times B) \times C.$$

Problem 11. Let G be any group. Prove that the map $G \to G$ taking $g \mapsto g^{-1}$ is a homomorphism if and only if G is abelian. What conditions guarantee that $g \mapsto g^{-1}$ is an automorphism?

Problem 12. Prove that D_8 and Q_8 are not isomorphic.

Problem 13. A *subgroup* of a group G is a subset $H \subseteq G$ such that
(1) $1 \in H$,
(2) if $a, b \in H$, then $ab \in H$, and
(3) if $a \in H$, then $a^{-1} \in H$.
Prove that H is a group under the operation $\cdot$ restricted to H.

Problem 14. Let G and H be groups and let $\varphi : G \to H$ be a homomorphism. Define the *kernel* of φ to be

$$\ker \varphi = \{g \in G \mid \varphi(g) = 1\}.$$

Prove that $\ker \varphi$ is a subgroup of G. Prove that φ is injective if and only if $\ker \varphi = \{1\}$.

Problem 15. Recall that Q_8 is the quaternion group of order 8 with generators i, j. Prove that the map φ from Q_8 to $GL_2(\mathbb{C})$ defined on generators by

$$\varphi(i) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \quad \text{and} \quad \varphi(j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

extends to a homomorphism. Prove that φ is in fact a monomorphism.