MATH 311: COMPLEX ANALYSIS
HOMEWORK DUE FRIDAY WEEK 1

Make sure to read the "Homework" portion of the syllabus before writing up your solutions!

Problem 1. Draw pictures of the sets of points $z \in \mathbb{C}$ satisfying the following relations:
(a) $|z - z_1| = |z - z_2|$ for some fixed $z_1 \neq z_2 \in \mathbb{C}$;
(b) $1/z = \overline{z}$;
(c) $|\mathrm{Im}(z)| \leq 1$.

Problem 2. Find all solutions in $\mathbb{C}$ to the equation $z^4 = -1$. Is the expression $\sqrt[4]{-1}$ unambiguous?

Problem 3. Let

$$\mathfrak{C} = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}$$

with the usual matrix addition and multiplication operations. Show that the function

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathfrak{C} \\ a + bi &\longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \end{aligned}$$

is an isomorphism of fields, where $a, b \in \mathbb{R}$. (You must check that f is a bijection and that it preserves both field operations.)

Problem 4. Show that it is impossible to define a total ordering on $\mathbb{C}$. In other words, there is no relation $<$ on $\mathbb{C}$ such that

(i) for any two $z, w \in \mathbb{C}$, precisely one of the following is true: $z < w$, $w < z$, $z = w$;
(ii) for all $z_1, z_2, z_3 \in \mathbb{C}$, if $z_1 < z_2$, then $z_1 + z_3 < z_2 + z_3$; and
(iii) for all $z_1, z_2, z_3 \in \mathbb{C}$ with $0 < z_3$, if $z_1 < z_2$, then $z_1 z_3 < z_2 z_3$.

[*Hint*: First consider what would happen if $0 < i$.]