Integration in $\mathbb{R}^n$

Following Colley, we have learned how to integrate functions $f : \mathbb{R}^n \to \mathbb{R}$ for $n = 2, 3$. There is, of course, nothing particularly special about these two natural numbers, except perhaps that it is easier to visualize and interpret double and triple integrals. In these notes, we will extend Colley's definitions to arbitrary $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and then comment briefly on appropriate extensions of the standard theorems. We conclude by computing the volume of the n-dimensional simplex of side r.

1. Definitions

Closed boxes (*aka* rectangles) in $\mathbb{R}^2$ take the form $[a_1, b_1] \times [a_2, b_2]$ where $a_i < b_i$ are real numbers for $i = 1, 2$. Closed boxes in $\mathbb{R}^3$ are of the form $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, again for $a_i < b_i$. In $\mathbb{R}^n$, a *closed box* is the cartesian product of n closed intervals:

$$B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

where $a_i < b_i$ for $i = 1, \ldots, n$. This is the same thing as the set of n-tuples $(x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $a_i \leq x_i \leq b_i$ for each $i = 1, \ldots, n$, and clearly generalizes the $n = 2, 3$ notions of closed box.

In order to integrate over such boxes, we need to partition them. A *partition* of B of order m consists of n collections of *partition points* that break up B into a union of m^n subboxes. That is, for $i = 1, \ldots, n$ we consider collections $\{x_{i,j}\}_{j=0}^{m}$ where

$$a_i = x_{i,0} < x_{i,1} < \ldots < x_{i,j-1} < x_{i,j} < \ldots < x_{i,m} = b_i.$$

Additionally, for $i = 1, \ldots, n$, $j = 1, \ldots, m$ we define

$$\Delta x_{i,j} = x_{i,j} - x_{i,j-1}.$$

Let $J = (j_1, j_2, \ldots, j_n)$ be a list of n indices j_i with $1 \leq j_i \leq m$. (Note that this is the same thing as J being an element of the n-fold cartesian product $\{1, 2, \ldots, m\}^n$.) The subboxes of this partition take the form

$$B_J = [x_{1,j_1}, x_{1,j_1-1}] \times [x_{2,j_2}, x_{2,j_2-1}] \times \cdots \times [x_{n,j_n}, x_{n,j_n-1}].$$

We define the *volume* of B_J to be

$$\Delta V_J = \Delta x_{1,j_1} \cdots \Delta x_{n,j_n},$$

the product of the widths of the n subintervals defining B_J.

Now let c_J be any point in the subbox B_J. For a function $f : B \to \mathbb{R}$, the quantity

$$S = \sum_J f(c_J) \Delta V_J,$$

where the indices J run through $\{1, 2, \ldots, m\}^n$, is called a *Riemann sum* of f on B corresponding to the partition.

This leads us to our primary definition, that of the *integral* (or *multiple integral*) of f on B; it is denoted

$$\int \cdots \int_B f \, dV \qquad \text{or} \qquad \int_B f \, dV$$

and is defined to be the limit of the Riemann sums S above in which all of the $\Delta x_{i,j}$ approach 0, provided this limit exists. (The first notation is meant to evoke n-many integral signs stacked next to each other. Since this is quite cumbersome, we will often use a single integral sign, remembering that the "multiplicity" of the integral is completely specified by the domain of f.) When $\int_B f \, dV$ exists, we say that f is *integrable* on B.

2. PROPERTIES

We are interested in describing some basic hypotheses under which a function $f : B \subseteq \mathbb{R}^n \to \mathbb{R}$ is integrable. The following is the basic result, presented without proof.

Theorem 1. *If f is bounded on B and the set of discontinuities of f has volume 0, then f is integrable on B.*

In order to understand this theorem, we must know what "volume 0" means for a subset of $\mathbb{R}^n$. The idea is the same as in $\mathbb{R}^2$ and $\mathbb{R}^3$: we can cover the set S of discontinuities with boxes which approach 0 in volume. But what is the volume of an n-dimensional box? We've already defined this above (*cf.* ΔV_J) as the product of the lengths of the intervals defining the box.

There is also an n-dimensional version of Fubini's theorem.

Theorem 2. *Let f be bounded on*

$$B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

and assume the set S of discontinuities of f has zero volume. If every line parallel to the coordinate axes of $\mathbb{R}^n$ meets S in finitely many points, then

$$\int_B f\, dV = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f(x_1, x_2, \ldots, x_n)\, dx_n \cdots dx_2\, dx_1.$$

If $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a bijection, then it is also the case that

$$\int_B f\, dV = \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} \int_{a_{\sigma(2)}}^{b_{\sigma(2)}} \cdots \int_{a_{\sigma(n)}}^{b_{\sigma(n)}} f(x_1, x_2, \ldots, x_n)\, dx_{\sigma(n)} \ldots dx_{\sigma(2)}\, dx_{\sigma(1)}.$$

Note that the final sentence in the theorem statement is just an elaborate (and precise!) way of saying that we can perform the iterated integral in any order. The bijection σ can be thought of as a "reordering" or *permutation* of $\{1, 2, \ldots, n\}$. There are $n!$ of these so, in some sense, Fubini's theorem in $\mathbb{R}^n$ is actually $n!$ different theorems.

3. INTEGRATING OVER GENERAL REGIONS

We would like to integrate over more general regions than just boxes in $\mathbb{R}^n$. Given $W \subseteq \mathbb{R}^n$ and $f : W \to \mathbb{R}$, we define f^{ext} to be the function on $\mathbb{R}^n$ given by

$$f^{ext}(x) = \begin{cases} f(x) & \text{if } x \in W, \\ 0 & \text{if } x \notin W. \end{cases}$$

If there is a box B such that $W \subseteq B$ (*i.e.* if W is bounded), then we define

$$\int_W f\, dV = \int_B f^{ext}\, dV,$$

provided the integral on the right exists. (It is readily checked that this definition does not depend on the choice of box B containing W.)

If f and g are integrable functions $W \to \mathbb{R}$, then $\int_W$ satisfies the usual linearity properties

$$\int_W (f + g)\, dV = \int_W f\, dV + \int_W g\, dV$$

and

$$\int_W cf\, dV = c \int_W f\, dV$$

for any constant $c \in \mathbb{R}$. Multiple integration is also monotonic (*i.e.*, respects inequalities), and

$$\left| \int_W f \, dV \right| \leq \int_W |f| \, dV.$$

Finally, we are left with the task of actually integrating functions over elementary regions. To say that $W \subseteq \mathbb{R}^n$ is *elementary* is to say that there is some reordering $y_1, \ldots, y_n$ of the coordinates $x_1, \ldots, x_n$ such that W consists of points $(x_1, \ldots, x_n)$ such that

$$\begin{gathered} a \leq y_1 \leq b, \\ \varphi_1^\ell(y_1) \leq y_2 \leq \varphi_1^h(y_1), \\ \varphi_2^\ell(y_1, y_2) \leq y_3 \leq \varphi_2^h(y_1, y_2), \\ \vdots \\ \varphi_{n-1}^\ell(y_1, \ldots, y_{n-1}) \leq y_n \leq \varphi_{n-1}^h(y_1, \ldots, y_{n-1}) \end{gathered}$$

where $a \leq b$ are constants and φ_i^ℓ and φ_i^h are functions of the indicated variables.

4. The n-dimensional simplex

For a positive integer n and nonnegative real number r, we define the *n-dimensional simplex of side r* to be

$$S_n(r) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \leq x_1, \ldots, 0 \leq x_n, x_1 + \cdots + x_n \leq r\}.$$

Take the time to draw pictures of $S_2(r)$ and $S_3(r)$. You should see that $S_2(r)$ is a solid right isosceles triangle in the first quadrant with side lengths r, r, and $\sqrt{2}r$. Meanwhile, $S_3(r)$ is a solid tetrahedron in the first octant.

From the definition, it is clear that $0 \leq x_n \leq r$ for all points $(x_1, \ldots, x_n) \in S_n(r)$. If we fix x_n between 0 and r and consider the set

$$S_n(r)|_{x_n} = \{(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \mid (x_1, \ldots, x_{n-1}, x_n) \in S_n(r)\},$$

then we see that

$$S_n(r)|_{x_n} = S_{n-1}(r - x_n).$$

This follows because

$$x_1 + \cdots + x_n \leq r \iff x_1 + \cdots + x_{n-1} \leq r - x_n.$$

We now compute the volume of $S_n(r)$ by first letting x_n vary from 0 to r. This gives us

$$\mathrm{vol}(S_n(r)) = \int_{S_n(r)} 1 \, dV = \int_0^r \left(\int_{S_{n-1}(r-x_n)} 1 \, dV \right) dx_n = \int_0^r \mathrm{vol}(S_{n-1}(r - x_n)) \, dx_n$$

because the "inner iterated integral" computes the volume of $S_{n-1}(r - x_n)$ by the above paragraph's observations. This sets us up perfectly to compute $\mathrm{vol}(S_n(r))$ inductively in n.

First observe that for any $r \geq 0$, $S_1(r) = [0, r]$ has volume (*i.e.*, length) r. Thus

$$\mathrm{vol}(S_2(r)) = \int_0^r \mathrm{vol}(S_1(r - x_2)) \, dx_2 = \int_0^r (r - x_2) \, dx_2 = rx_2 - \frac{x_2^2}{2} \bigg|_0^r = \frac{r^2}{2}.$$

Based on this (and perhaps a few more low-dimensional calculations), we guess that

$$\mathrm{vol}(S_n(r)) = \frac{r^n}{n!}.$$

We have already verified the base case (when $n = 1$), so suppose that for any $r \geq 0$ and some $n > 1$ we have $\mathrm{vol}(S_{n-1}(r)) = r^{n-1}/(n-1)!$. Then

$$\mathrm{vol}(S_n(r)) = \int_0^r \mathrm{vol}(S_{n-1}(r - x_n))\, dx_n = \int_0^r \frac{(r - x_n)^{n-1}}{(n-1)!}\, dx_n.$$

Let $u = r - x_n$ so that $du = -dx_n$. Then we get

$$\mathrm{vol}(S_n(r)) = \frac{-1}{(n-1)!} \int_r^0 u^{n-1}\, du = \frac{-1}{(n-1)!} \left. \frac{u^n}{n} \right|_r^0 = \frac{r^n}{n!},$$

as desired, completing our proof by induction.