WEEK 12 HOMEWORK

MATH 211

Problem 1. Consider the vector fields $F, G, H : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$F(x, y, z) = (x, 2z, -z^2), \quad G(x, y, z) = (y, -x, z), \quad H(x, y, z) = (2, -3y, z^3)$$

and the paths $\gamma, \delta, \varphi : \mathbb{R} \to \mathbb{R}^3$ given by

$$\gamma(t) = (\sin t, \cos t, e^t), \quad \delta(t) = (e^t, 2 \log t, 1/t), \quad \varphi(t) = (2t + 3, 5e^{-3t}, 7/\sqrt{1 - 98t}).$$

Determine which paths are flow lines for which vector fields, explaining your work.

Problem 2. For $\alpha \in \mathbb{R}$, let $L : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation taking x to αx. Compute the operator norm of L, proving that your answer is correct.

Problem 3. Prove Proposition 8 from the notes; *i.e.* prove that for any linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$,

$$\begin{aligned} ||L|| &= \sup\{|L(v)| \mid |v| \leq 1\} \\ &= \sup\{|L(v)| \mid |v| = 1\} \\ &= \sup\{|L(v)|/|v| \mid v \in \mathbb{R}^n \setminus \{0\}\}. \end{aligned}$$

Problem 4. Prove part (ii) of Proposition 9 from the notes; *i.e.*, prove that for any linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ and any scalar $\lambda \in \mathbb{R}$,

$$||\lambda L|| = |\lambda| \cdot ||L||.$$

Problem 5. Let A be the 3×3 matrix

$$A = \begin{pmatrix} 4 & 2 & -8 \\ 3 & 0 & -6 \\ 2 & 1 & -4 \end{pmatrix}.$$

(a) Compute a closed form for the exponential matrix e^{At} where t is some scalar. (*Hint*: $A^3 = 0$.)
(b) Write out the homogeneous linear system of differential equations given by

$$x' = Ax.$$

(c) Use e^{At} to solve the above system of differential equations subject to the initial condition $x(0) = (1, 1, 1)$.
(d) Use `sage` or another computer algebra system to plot the vector field Ax and the flow line of your solution, $x(t)$.

Problem 6. Let B be the 2×2 matrix

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(a) Show that $B^{2n} = I$ and $B^{2n+1} = B$ for all natural numbers n.
(b) Use (a) to show that

$$e^{Bt} = \cosh(t)I + \sinh(t)B.$$

(You may take the Taylor series for hyperbolic sine and cosine as given.)

(c) Plot a family of flow lines for the differential equation

$$\gamma'(t) = B\gamma(t)$$

by varying initial conditions. Describe their behavior in qualitative terms.

Problem 7. Repeat the analysis of Problem 6 but with B replaced by the matrix

$$C = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

(The functions from part (b) will no longer be hyperbolic trig functions, but they should be familiar nonetheless.)

Problem 8 (Bonus). The *Frobenius norm* (also known as the Hanson norm) of an $m \times n$ matrix $A = (a_{ij})$ is

$$|A|_2 = \sqrt{\sum_{1 \le i \le m} \sum_{1 \le j \le n} a_{ij}^2}.$$

(It is given by considering the mn entries in A as forming a vector in $\mathbb{R}^{mn}$ and then taking the Euclidean norm.) Prove that the Frobenius norm dominates the operator norm, *i.e.*, that

$$||A|| \le |A|_2$$

for all matrices A. (Partial bonus credit will be given for proving this in the 2×2 case.)