Day 33

Learning Goals

- Determinants and volume
- Determinants and orientation

Determinants redux

What good is det?

① $A \in \mathrm{Mat}_{n\times n}(F)$ is invertible iff $\det A \neq 0$

② $\chi_A(x) = \det(A - xI_n)$ has roots = eigenvalues of A.

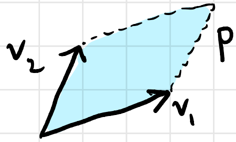
Viewing det in the context of real inner product spaces, we get two more applications:

③ A scales volume by $|\det A|$

④ sign $\det A$ measures whether A is orientation-preserving or -reversing.

Volume

Defn The *parallelepiped* spanned by $v_1, \dots, v_n \in \mathbb{R}^n$ is the set $P := \left\{ \sum_{i=1}^{n} \lambda_i v_i \;\middle|\; 0 \leq \lambda_i \leq n \;\forall i \right\}$.



Thm Suppose $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ has rows $v_1, \dots, v_n \in \mathbb{R}^n$ and that P is the parallelipiped spanned by $v_1, \dots, v_n$. Then $\mathrm{vol}(P) = |\det A|$.

Remarks ① Volume? We will argue for the reasonableness of this theorem. A full-fledged discussion of volume occurs in Math 202 + 321.

② For $n=1$, vol = length; $n=2$, vol = area.

③ If $v_1, \dots, v_n$ are lin ind, then $\det A = 0$. In this case, P lives in a proper subspace of $\mathbb{R}^n$ which has 0 n-dim'l volume.

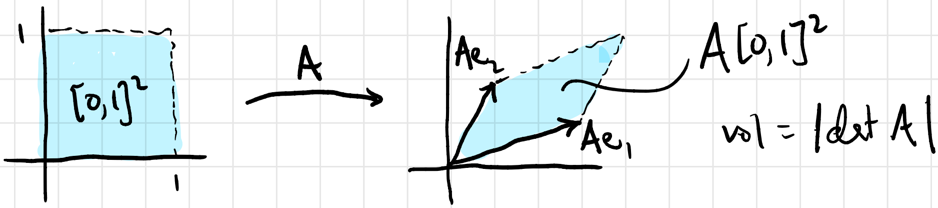
④ Intuitive properties of volume captured by this theorem

(a) If we scale one v_i by $\lambda \geq 0$, then the vol(P) scales by λ. If we scale all v_i by $\lambda \geq 0$, then vol(P) scales by λ^n.

(b) If we permute the v_i, vol(P) does not change.

⑤ $\det A = \det A^T$, so $|\det A| = \mathrm{vol}$(parallelipiped spanned by columns of A). This is more geometrically meaningful, since this says

$$\boxed{|\det A| = \mathrm{vol}\big(\underbrace{A\cdot[0,1]^n}_{\text{image of unit cube under } A}\big)}.$$



"Pf" of Thm Apply Gram-Schmidt to get orthogonal vectors $\{\tilde{v}_1, \dots, \tilde{v}_n\}$ with

$$
\begin{aligned}
v_1 &= \tilde{v}_1 \\
v_2 &= a_{21}\tilde{v}_1 + \tilde{v}_2 \\
v_3 &= a_{31}\tilde{v}_1 + a_{32}\tilde{v}_2 + \tilde{v}_3 \\
&\vdots \\
v_n &= a_{n1}\tilde{v}_1 + a_{n2}\tilde{v}_2 + \cdots + a_{n,n-1}\tilde{v}_{n-1} + \tilde{v}_n .
\end{aligned}
$$

Since det is alternating multilinear in rows,

$$\begin{aligned}\det A &= \det(v_1, \dots, v_n) \\ &= \det(\tilde{v}_1, a_{21}\tilde{v}_1 + \tilde{v}_2, a_{31}\tilde{v}_1 + a_{32}\tilde{v}_2 + \tilde{v}_3, \dots) \\ &= \det(\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_n).\end{aligned}$$

Let $\tilde{A}$ have rows $\tilde{v}_1, \dots, \tilde{v}_n$. We just showed $\det A = \det \tilde{A}$.

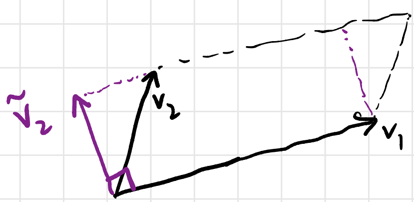
Thus
$$\begin{aligned}\det(A)^2 &= \det(A)\det(A^T) \\ &= \det(\tilde{A})\det(\tilde{A}^T) \\ &= \det(\tilde{A}\tilde{A}^T).\end{aligned}$$

But $\tilde{v}_1, \dots, \tilde{v}_n$ are orthogonal, so $\tilde{A}\tilde{A}^T = \begin{pmatrix} \|\tilde{v}_1\|^2 & & 0 \\ & \ddots & \\ 0 & & \|\tilde{v}_n\|^2 \end{pmatrix}$

$$\Longrightarrow \det(A)^2 = \|\tilde{v}_1\|^2 \cdots \|\tilde{v}_n\|^2$$

$$\Longrightarrow |\det A| = \|\tilde{v}_1\| \cdots \|\tilde{v}_n\|.$$

We claim the RHS is the volume of P:



This is the base · height · width · $\cdots$ formula for volume of a parallelepiped! $\square$

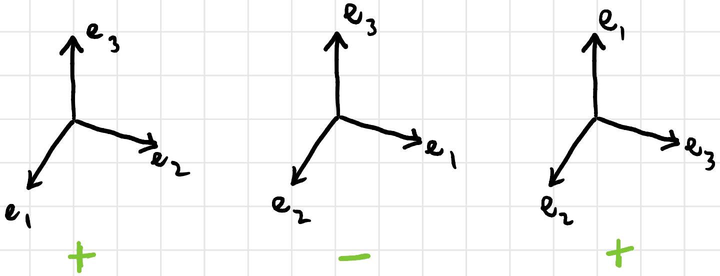
Orientation

Recall that swapping two rows of a matrix multiplies its determinant by -1.

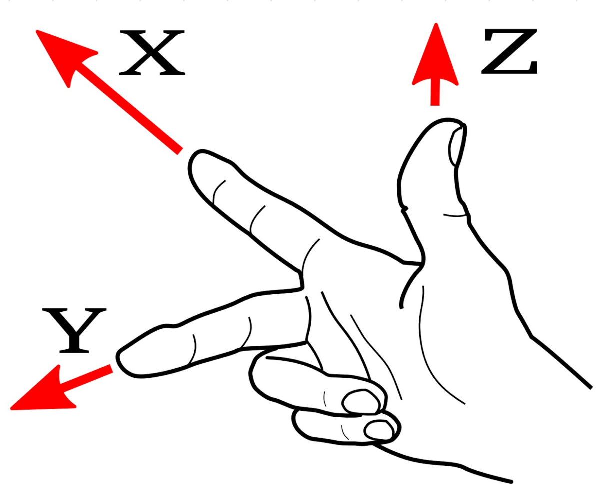
Defn The orientation of an ordered list of lin ind vectors $v_1, \dots, v_n \in \mathbb{R}^n$ is the sign of $\det(v_1, \dots, v_n)$.

E.g.

- $e_1, e_2, e_3 \in \mathbb{R}^3$ has positive orientation.
- Swapping e_1, e_2 gives e_2, e_1, e_3 with negative orientation.
- Swapping again to get e_2, e_3, e_1 gives positive orientation.



- Note that the positive orientations obey the "right-hand rule."



Volume again

Let's return to the fact

$$|\det A| = \mathrm{vol}(A[0,1]^n).$$

Let P be a parallelipiped spanned by $v_1, \dots, v_n$ and M the matrix with columns $v_1, \dots, v_n$ so that $|\det M| = \mathrm{vol}(P)$.

What about $\mathrm{vol}(AP)$? We have

$$AP = AM[0,1]^n \Rightarrow \mathrm{vol}(AP) = |\det(AM)|$$
$$= |\det A||\det M|$$
$$= |\det A|\,\mathrm{vol}(P)$$

so $\dfrac{\operatorname{vol}(AP)}{\operatorname{vol}(P)} = |\det A|$ — i.e. volumes of all parallelipipeds are scaled by $|\det A|$ when transformed by A.

Note further that volume is not altered by translation:

$$\begin{aligned}\operatorname{vol}(A(v+P)) &= \operatorname{vol}(Av + AP)\\ &= \operatorname{vol}(AP)\\ &= |\det A| \operatorname{vol}(v+P).\end{aligned}$$

Now suppose $X \subseteq \mathbb{R}^n$ is "nice" in the sense that it may be "approximated" by parallelipipeds.

Then $\operatorname{vol}(AX) = |\det A| \operatorname{vol}(X)$

i.e. $|\det A|$ is a universal scaling factor for volume after applying A.

Change of variables

For $X \subseteq \mathbb{R}^n$ "nice" and $\rho : \mathbb{R}^n \to \mathbb{R}$ "nice",

$$\int_X \rho \in \mathbb{R}$$

is the "volume of X weighted by ρ."

(You will define this properly in Math 202.)

Let f be a "change of coordinates" on X.

Then the change of variables formula says

$$\int_{f(X)} \rho = \int_X (\rho \circ f)\underbrace{|\det \underbrace{f'}_{\substack{\text{derivative of } f \\ = \text{ best linear approximation}}}|}_{\substack{\text{measure of how} \\ \text{much } f \text{ locally stretches space!}}}.$$