Day 32

Learning Goals

- Direct sum
- Orthogonal complement
- Orthogonal projection
- Least squares

Defn Given F-vector spaces U, V, the direct sum of U and V is $U \oplus V := U \times V = \{(u,v) \mid u \in U, v \in V\}$ with componentwise add'n and scalar mult'n:

$$(u,v) + (u',v') = (u+u', v+v')$$

$$\lambda(u,v) = (\lambda u, \lambda v).$$

Prop Let $U, V \leq W$ s.t.

① $\operatorname{span}(U \cup V) = W$

② $U \cap V = \{0\}$.

Then $U \oplus V \longrightarrow W$ is an isomorphism.
$(u,v) \longmapsto u+v$

Pf Linearity is clear. ① guarantees surjectivity. If $u+v=0$ for $u \in U, v \in V$ then $u = -v \in V \cap U \Rightarrow u = v = 0$ by ②.

Thus $\ker = \{0\}$ so the map is injective as well. $\square$

From here on, let $(V, \langle , \rangle)$ be an IPS over $F = \mathbb{R}$ or $\mathbb{C}$.

Defn For $S \subseteq V$, the **orthogonal complement** of S is
$$S^\perp := \{x \in V \mid \langle x, s\rangle = 0 \quad \forall s \in S\}.$$

TPS $S^\perp$ is a subspace of V.

Prop Suppose $\dim V = n$ and $S = \{v_1, \dots, v_k\} \subseteq V$ is orthonormal.

1. S can be extended to an orthonormal basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ of V.
2. If $W = \operatorname{span} S$, then $\{v_{k+1}, \dots, v_n\}$ is an orthonormal basis for $W^\perp$.
3. If $W \leq V$, then $\dim W + \dim W^\perp = \dim V = n$.
4. If $W \leq V$, then $(W^\perp)^\perp = W$.

Pf 1. Apply Gram-Schmidt starting with $\{v_1, \dots, v_k\}$.

(2) Let $S' = \{v_{k+1}, \dots, v_n\}$ which is lin ind and orthonormal. Since S is orthonormal,

$S' \subseteq W^\perp \Rightarrow \text{span } S' \subseteq W^\perp$.

For $x \in W^\perp$, since S is orthonormal,

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$$

$$= \sum_{i=k+1}^{n} \langle x, v_i \rangle v_i \qquad (x \in W^\perp)$$

$$\in \text{span } S'$$

so $\text{span } S' = W^\perp \Rightarrow S'$ a basis for $W^\perp$.

(3) Let $S = \{v_1, \dots, v_k\}$ be an orthonormal basis for W and let $S' = \{v_{k+1}, \dots, v_n\}$ be as in (2).

Then $\dim W = k$, $\dim W^\perp = n-k$.

(4) We have $(W^\perp)^\perp = \{x \in V \mid \langle x, y \rangle = 0 \text{ for } y \in W^\perp\}$

$$\supseteq W.$$

Now $\dim (W^\perp)^\perp = n - \dim W^\perp = \dim W$

so $W = (W^\perp)^\perp$. $\square$

Prop Let $W \leq V$. Then $V = W \oplus W^\perp$, i.e. for $y \in V$ $\exists!$ $u \in W$, $v \in W^\perp$ s.t.

$$y = u + v.$$

Call u the orthogonal projection of y onto W.
If $\{u_1, \dots, u_k\}$ is an orthonormal basis for W, then $u = \sum_{i=1}^{k} \langle y, u_i \rangle u_i$.

Pf Let $\{u_1, \dots, u_k\}$ be an orthonormal basis of W and define $u = \sum_{i=1}^{k} \langle y, u_i \rangle u_i$, $v = y - u$. Then $u \in W$ and $y = u + v$. Furthermore, for $1 \leq j \leq k$,

$$\begin{aligned}
\langle v, u_j \rangle &= \langle y - u, u_j \rangle \\
&= \langle y, u_j \rangle - \left\langle \sum_{i=1}^{k} \langle y, u_i \rangle u_i, u_j \right\rangle \\
&= \langle y, u_j \rangle - \sum_{i=1}^{k} \langle y, u_i \rangle \langle u_i, u_j \rangle \\
&= \langle y, u_j \rangle - \langle y, u_j \rangle \cdot 1 \\
&= 0
\end{aligned}$$

so $v \in W^\perp$.

Moral exc $W \cap W^\perp = \{0\}$ and the expression is unique. $\square$

Cor The orthogonal projection u of y onto W is the vector in W closest to y:

$$\|y-u\| \leq \|y-w\|$$

for all $w \in W$ with equality iff $u=w$.

Pf Write $y = u+v$ with $u \in W$, $v \in W^\perp$. Take $w \in W$. Then $u-w \in W$, $y-u \in W^\perp$. By Pythagoras,

$$\begin{aligned}\|y-w\|^2 &= \|(y-u)+(u-w)\|^2 \\ &= \|y-u\|^2 + \|u-w\|^2 \\ &\geq \|y-u\|^2\end{aligned}$$

with equality iff $\|u-w\|=0$ iff $u=w$. $\square$

E.g. Let $W = \text{span}\{\underset{\vec{u}_1}{(1,1,0)}, \underset{\vec{u}_2}{(0,1,1)}\}$. We determine the (minimal) distance of $\overset{\vec{y}}{(4,0,-1)}$ from W:

The orthogonal proj'n of y onto W is

$$u = \frac{\langle y, u_1 \rangle}{\|u_1\|^2} u_1 + \frac{\langle y, u_2 \rangle}{\|u_2\|^2} u_2$$

$$= (3, 1, -2)$$

so $y - u = (1, -1, 1)$ with $\|y - u\| = \sqrt{3}$ the distance of y from W.

Least squares

Goal Given $A \in \mathrm{Mat}_{m \times n}(\mathbb{R})$, $b = \mathbb{R}^m$, find $x \in \mathbb{R}^n$ s.t. $\|r(x) := Ax - b\|$ is as small as possible.

— call this $\hat{x}$, the least squares solution.

Thm A vector $\hat{x} \in \mathbb{R}^n$ is the least squares solution of $Ax = b$ iff it is a sol'n of the associated normal system $A^T A x = A^T b$.

Pf Idea $r(x)$ is minimized when it is orthogonal to the row space of A.

Show that $\mathrm{row}(A)^\perp = \ker(A^T)$ so $\hat{x}$ satisfies

$$A^T r(\hat{x}) = 0 \iff A^T(b - A\hat{x}) = 0$$

$$\iff A^T A \hat{x} = A^T b.$$

Fact The normal system $A^TAx = A^Tb$ is consistent $\Longleftrightarrow$ $\ker(A) = 0$ $\Longleftrightarrow$ columns of A are lin ind.