Day 31

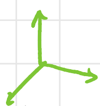
Learning goals

- Orthonormal vectors
- Gram-Schmidt orthonormalization algorithm

Throughout, $(V, \langle , \rangle)$ an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Defn For $S \subseteq V$, S is *orthogonal* if $\langle u, v \rangle = 0$ for all $u \neq v \in S$; it is *orthonormal* when additionally $\langle u, u \rangle = 1$ for all $u \in S$.

E.g. (1) The standard basis $\{e_1, \ldots, e_n\}$ is orthonormal in $(\mathbb{F}^n, \text{ordinary})$.



(2) The set $\{\frac{1}{\sqrt{2}}(1,1), \frac{1}{\sqrt{2}}(1,-1)\}$ is orthonormal in $(\mathbb{R}^2, \text{ordinary})$.

Prop Let $S = \{v_1, \ldots, v_k\} \subseteq V$ be orthogonal. Then for $y \in \text{span}(S)$,

$$y = \sum_{j=1}^{k} \frac{\langle y, v_j \rangle}{\|v_j\|^2} v_j$$

projections of y onto v_j

Pf Say $y = \sum_{i=1}^{k} \lambda_i v_i$. Then

$$\langle y, v_j \rangle = \left\langle \sum_i \lambda_i v_i , v_j \right\rangle$$

$$= \sum_i \lambda_i \langle v_i , v_j \rangle$$

$$= \lambda_j \langle v_j , v_j \rangle \qquad (\text{since } \langle v_i, v_j \rangle = 0 \text{ for } i \neq j)$$

$$= \lambda_j \|v_j\|^2 .$$

Thus $\lambda_j = \dfrac{\langle y, v_j \rangle}{\|v_j\|^2}$ is the component of y along v_j. $\square$

Cor If $S = \{v_1, \dots, v_k\} \subseteq V$ is orthonormal and $y \in \operatorname{span}(S)$, then $y = \sum_{j=1}^{k} \langle y, v_j \rangle v_j$. $\square$

Cor If $S \subseteq V$ is an orthogonal set of vectors, then S is linearly independent.

Pf Let $S = \{v_1, \dots, v_k\}$ and suppose $\sum_i \lambda_i v_i = 0$.

Then $0 = \langle 0, v_j \rangle = \left\langle \sum_i \lambda_i v_i , v_j \right\rangle = \sum_i \lambda_i \langle v_i, v_j \rangle$

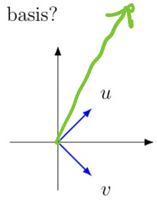
$$= \lambda_j \langle v_j, v_j \rangle$$

$\neq 0$ so $\lambda_j = 0$. $\square$

Example. Consider $\mathbb{R}^2$ with the standard inner product, and let

$$u = \frac{1}{\sqrt{2}}(1,1) \quad \text{and} \quad v = \frac{1}{\sqrt{2}}(1,-1).$$

Then $\beta = \{u, v\}$ gives an orthonormal ordered basis for $\mathbb{R}^2$. What are the coordinates of $y = (4,7)$ with respect to that basis?



Answer:

$$\begin{aligned} y &= \langle y, u\rangle u + \langle y, v\rangle v \\ &= (4,7)\cdot\left(\frac{1}{\sqrt{2}}(1,1)\right)u + (4,7)\cdot\left(\frac{1}{\sqrt{2}}(1,-1)\right)v \\ &= \frac{11}{\sqrt{2}}u - \frac{3}{\sqrt{2}}v. \end{aligned}$$

Check:

$$\frac{11}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(1,1)\right) - \frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(1,-1)\right) = \frac{11}{2}(1,1) - \frac{3}{2}(1,-1) = (4,7).$$

Goal Turn a linearly independent set into an orthonormal set.

Algorithm [Gram-Schmidt]

Input: Lin ind set $S = \{w_1, \dots, w_n\} \subseteq V$.

(1) Let $v_1 = w_1$.

$(2, \dots, n)$ For $k = 2, \dots, n$, define

$$v_k = w_k - \sum_{i=1}^{k-1} \frac{\langle w_k, v_i\rangle}{\|v_i\|^2} v_i$$

Output: $S' = \{v_1, \dots, v_n\} \subseteq V$ an orthogonal set with span S' = span S.

or

Output: $S'' = \left\{ \frac{v_1}{\|v_1\|}, \dots, \frac{v_n}{\|v_n\|} \right\}$ an orthonormal set with span S'' = span S.

Think of this as "straightening" w_k wrt $v_1, \dots, v_{k-1}$ by removing proj'ns onto them

Validity Pf Idea Use induction to check that the "straightening operation" preserves orthogonality and span. (Full details below.) $\square$

Cor Every finite dim'l inner product space has an orthonormal basis.

Pf Apply Gram-Schmidt to any basis. $\square$

E.g. Take $V = \mathbb{R}[x]_{\leq 1}$ with inner product $\langle f, g \rangle = \int_0^1 fg$.

Let's apply G-S to $\{1, x\}$:

(1) $v_1 = 1$

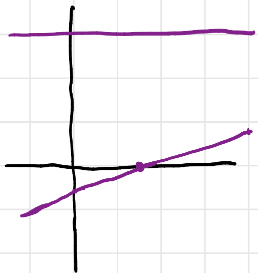
(2) $v_2 = x - \frac{\langle x, v_1 \rangle}{\|v_1\|^2} v_1 = x - \frac{\langle x, 1 \rangle}{\|1\|^2} 1$

$$= x - \int_0^1 x\,dx = x - \frac{1}{2}.$$

Normalizing: $\|v_1\| = \sqrt{\int_0^1 1^2 dx} = 1$

$\|v_2\| = \sqrt{\int_0^1 (x - 1/2)^2 dx} = \sqrt{1/12}$

So $\{1, \sqrt{12}(x - 1/2)\}$ is an orthonormal basis.



TPS How can we interpret orthonormality of these visually?

TPS Orthonormalize $\{(1,2),(0,-1)\}$ in $\mathbb{R}^2$.

Proof of validity of the algorithm. We prove this by induction on n. The case $n = 1$ is clear. Suppose the algorithm works for some $n \geq 1$. and let $S = \{w_1, \ldots, w_{n+1}\}$ be a linearly independent set. By induction, running the algorithm on the first n vectors in S produces orthogonal $v_1, \ldots, v_n$ with

$$\operatorname{Span}\{v_1, \ldots, v_n\} = \operatorname{Span}\{w_1, \ldots, w_n\}.$$

Running the algorithm further produces

$$v_{n+1} = w_{n+1} - \sum_{i=1}^{n} \frac{\langle w_{n+1}, v_i \rangle}{\|v_i\|^2} v_i.$$

It cannot be that $v_{n+1} = 0$, since otherwise, the above equation we would say

$$w_{n+1} \in \operatorname{Span}\{v_1, \ldots, v_n\} = \operatorname{Span}\{w_1, \ldots, w_n\},$$

contradicting the assumption of the linear independence of the w_i. So $v_{n+1} \neq 0$.

We now check that v_{n+1} is orthogonal to the previous v_i. For $j = 1, \ldots, n$, we have

$$\begin{aligned}
\langle v_{n+1}, v_j \rangle &= \left\langle w_{n+1} - \sum_{i=1}^{n} \frac{\langle w_{n+1}, v_i \rangle}{\|v_i\|^2} v_i, v_j \right\rangle \\
&= \langle w_{n+1}, v_j \rangle - \sum_{i=1}^{n} \frac{\langle w_{n+1}, v_i \rangle}{\|v_i\|^2} \langle v_i, v_j \rangle \\
&= \langle w_{n+1}, v_j \rangle - \frac{\langle w_{n+1}, v_j \rangle}{\|v_j\|^2} \langle v_j, v_j \rangle \\
&= \langle w_{n+1}, v_j \rangle - \langle w_{n+1}, v_j \rangle \\
&= 0.
\end{aligned}$$

We have shown $\{v_1, \ldots, v_{n+1}\}$ is an orthogonal set of vectors, and we would now like to show that its span is the span of $\{w_1, \ldots, w_{n+1}\}$. First, since $\{v_1, \ldots, v_{n+1}\}$ is orthogonal, it's linearly independent. Next, we have

$$\operatorname{Span}\{v_1, \ldots, v_{n+1}\} \subseteq \operatorname{Span}\{v_1, \ldots, v_n, w_{n+1}\} \subseteq \operatorname{Span}\{w_1, \ldots, w_n, w_{n+1}\}.$$

Since $\operatorname{Span}\{v_1, \ldots, v_{n+1}\}$ is an $(n+1)$-dimensional subspace of the $(n+1)$-dimensional space $\operatorname{Span}\{w_1, \ldots, w_n, w_{n+1}\}$, these spaces must be equal. $\square$