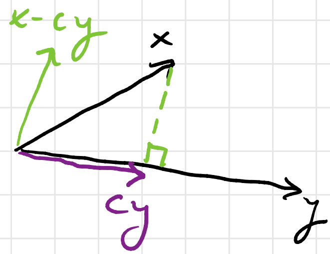
Day 30

Learning Goals

- Components + (orthogonal) projection
- Cauchy-Schwarz & triangle inequalities
- Angles between vectors

Throughout, $(V, \langle , \rangle)$ an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$. Recall $x, y \in V$ are orthogonal when $\langle x, y \rangle = 0$.

Provocation:



Given $x, y \in V$, can we find $c \in F$ such that $\langle x - cy, y \rangle = 0$?

Answer: $\langle x-cy, y \rangle = 0 \iff \langle x, y \rangle - c\langle y, y \rangle = 0$

$$\iff c\langle y, y \rangle = \langle x, y \rangle$$

$$\iff c = \frac{\langle x, y \rangle}{\langle y, y \rangle} = \frac{\langle x, y \rangle}{\|y\|^2}$$

for $y \neq 0$

Defn The component of x along y is is

$$c = \frac{\langle x, y\rangle}{\|y\|^2}$$

and the (orthogonal) projection of x to y is the vector $cy = \frac{\langle x, y\rangle}{\|y\|^2} y$.

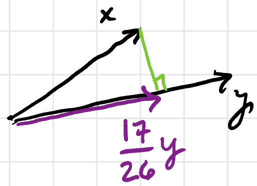
E.g. (1) $V = F^n$ w/ ordinary inner product, e_j = j-th standard basis vector. Then for $x \in V$,

$$\frac{\langle x, e_j\rangle}{\|e_j\|^2} = x_j,$$ the j-th component of $x = (x_1, \dots, x_n)$.

The projection of x to e_j is $x_j e_j$.

(2) $x = (3, 2)$, $y = (5, 1)$ in $\mathbb{R}^2$ w/ ordinary IP.

Then $$\frac{\langle x, y\rangle}{\|y\|^2} = \frac{3\cdot 5 + 2\cdot 1}{5\cdot 5 + 1\cdot 1} = \frac{17}{26} \approx 0.65.$$



Thm For $x, y \in V$, $\lambda \in F$,

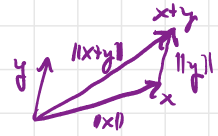
$|\;|$: absolute value on $\mathbb{R}$ or length in $\mathbb{C}$: $|a+bi| = \sqrt{a^2+b^2}$.

(1) $\|\lambda x\| = |\lambda| \, \|x\|$

(2) $\|x\| = 0 \iff x = 0$

(3) **Cauchy-Schwarz inequality**: $|\langle x, y\rangle| \le \|x\| \, \|y\|$

(4) **Triangle inequality**: $\|x+y\| \le \|x\| + \|y\|$



Pf (1), (2): Moral exercises.

(3): If $y = 0$, we're done as $0 \le 0$. For $y \ne 0$, let $c = \dfrac{\langle x, y\rangle}{\|y\|^2}$. Then $x - cy$ is orthogonal to y, hence orthogonal to cy. By Pythagoras,

$$\|x - cy\|^2 + \|cy\|^2 = \|x - cy + cy\|^2 = \|x\|^2.$$

Since $\|x-cy\|^2 \ge 0$, get $\|cy\|^2 \le \|x\|^2$. Taking square roots,

$$\|x\| \ge \|cy\| = |c| \, \|y\| = \left| \frac{\langle x, y\rangle}{\|y\|^2} \right| \|y\| = \left| \frac{\langle x, y\rangle}{\|y\|} \right|.$$

Thus $\|x\| \, \|y\| \ge |\langle x, y\rangle|$. ✓

(4) First "recall" the following about complex numbers:

(i) $z + \bar{z} = 2\operatorname{Re}(z)$ (ii) $\operatorname{Re}(z) \le |z|$.

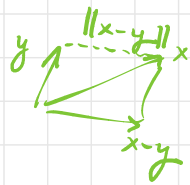
Now observe that

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \langle x,x\rangle + \langle y,y\rangle + \langle x,y\rangle + \langle y,x\rangle$$
$$= \|x\|^2 + \|y\|^2 + \langle x,y\rangle + \overline{\langle x,y\rangle} \qquad \text{(conj symm)}$$
$$= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(\langle x,y\rangle) \qquad \text{(i)}$$
$$\leq \|x\|^2 + \|y\|^2 + 2|\langle x,y\rangle| \qquad \text{(ii)}$$
$$\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \qquad \text{(Cauchy-Schwarz)}$$
$$= (\|x\| + \|y\|)^2.$$

Taking square roots gives the Δ ineq. ✓ ☐

Aside Define the **distance** b/w $x, y \in V$ to be

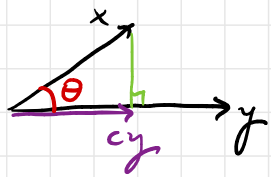
$$d(x,y) := \|x-y\|.$$



This makes (V,d) a **metric space**:

- $d(x,y) = d(y,x)$
- $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$
- $d(x,z) \leq d(x,y) + d(y,z)$.

Angles



Idea

$$\cos\theta = \frac{\|cy\|}{\|x\|} = |c|\frac{\|y\|}{\|x\|}$$

$$= \frac{|\langle x,y\rangle|}{\|y\|^2}\,\frac{\|y\|}{\|x\|}$$

$$= \frac{|\langle x,y\rangle|}{\|x\|\,\|y\|}.$$

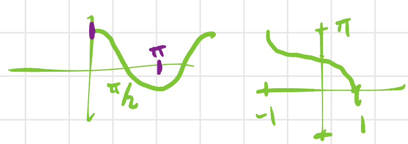
Dropping the $|\ |$ in the numerator makes this work in all quadrants. So...

Dfn Let $(V, \langle\,,\rangle)$ be an IPS over $F=\mathbb{R}$. The **angle** θ between $x,y\in V$ is

$$\theta := \arccos\left(\frac{\langle x,y\rangle}{\|x\|\,\|y\|}\right)$$

I.e. $\cos\theta = \dfrac{\langle x,y\rangle}{\|x\|\,\|y\|}$ and $\langle x,y\rangle = \|x\|\,\|y\|\cos\theta$.

Note. By Cauchy-Schwarz,



arccos

$$\frac{|\langle x,y\rangle|}{\|x\|\,\|y\|} \leq 1 \quad \text{so} \quad -1 \leq \frac{\langle x,y\rangle}{\|x\|\,\|y\|} \leq 1$$

and thus arccos of makes sense.

- Can also write

$$\theta = \arccos\left(\left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle\right).$$

$\frac{v}{\|v\|}$ is the unit vector in the direction of v.