Day 29

Learning Goals

- Define inner product spaces (IPS)
- Norm / length in IPS

Motivation Add structure to $\mathbb{R}$- or $\mathbb{C}$-vector spaces so that we can define length and angles.

Defn Let $F = \mathbb{R}$ or $\mathbb{C}$ and let V be an F-v.s.

An inner product on V is a function

$$\langle , \rangle : V \times V \longrightarrow F$$
$$(x, y) \longmapsto \langle x, y \rangle$$

s.t. $\forall x, y, z \in V$, $\lambda \in F$

① linearity: $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ and $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$

② conjugate symmetry: $\overline{\langle x, y \rangle} = \langle y, x \rangle$

③ positive definiteness: $\langle x, x \rangle \in \mathbb{R}_{\geq 0}$ and $\langle x, x \rangle = 0$ iff $x = 0$.

Note $F = \mathbb{R}$: this is a non-degenerate positive definite symmetric bilinear form

$F = \mathbb{C}$: this is a non-degenerate Hermitian form

E.g. (a) Ordinary dot product on $V = \mathbb{R}^n$:

$$\langle (x_1,\dots,x_n),(y_1,\dots,y_n)\rangle$$
$$= x\cdot y := \sum_{i=1}^{n} x_i y_i = x_1y_1+\cdots+x_ny_n.$$

e.g. $\langle (1,2),(3,4)\rangle = 1\cdot 3+2\cdot 4 = 11.$

(b) Ordinary inner product on $V=\mathbb{C}^n$:

$$\langle (x_1,\dots,x_n),(y_1,\dots,y_n)\rangle$$
$$= x\cdot\overline{y} := \sum_{i=1}^{n} x_i\overline{y_i} = x_1\overline{y_1}+\cdots+x_n\overline{y_n}.$$

e.g. $\langle (1+i,1-i),(1+2i,4)\rangle$

$$= (1+i)\overline{(1+2i)} + (1-i)\overline{4}$$

$$= 7-5i.$$

(c) Let $V = C_{\mathbb{R}}([0,1]) = \{f:[0,1]\to\mathbb{R} \mid f \text{ is cts}\}$,

$$\langle f,g\rangle := \int_0^1 f(t)\,g(t)\,dt.$$

For pos def, note $f\neq 0 \Rightarrow f(t)^2 \geq 0$ and $f(t)^2>0$ on some open interval $\Rightarrow \langle f,f\rangle = \int_0^1 f(t)^2\,dt > 0.$

(d) $V=\mathbb{R}^2$ and

$$\langle (x_1,x_2),(y_1,y_2)\rangle = 3x_1y_1+2x_1y_2+2x_2y_1+4x_2y_2.$$

For pos def, note that

$$\langle (x_1,x_2),(x_1,x_2)\rangle = 3x_1^2+4x_1x_2+4x_2^2$$

$$= 3\left(x_1^2 + \tfrac{4}{3}x_1x_2 + \tfrac{4}{3}x_2^2\right)$$

$$= 3\left(\left(x_1 + \tfrac{2}{3}x_2\right)^2 - \tfrac{4}{9}x_2^2 + \tfrac{4}{3}x_2^2\right)$$

$$= 3\left(\left(x_1 + \tfrac{2}{3}x_2\right)^2 + \tfrac{8}{9}x_2^2\right)$$

≥ 0, with $=$ iff $x_1 = x_2 = 0$.

(2) $V = \mathrm{Mat}_{m\times n}(F)$. For $A \in V$, define the conjugate transpose of A by $A^* := \overline{A}^T$ with $A^*_{ij} = \overline{A_{ji}}$.

Then $\langle A, B\rangle := \mathrm{tr}(B^*A) = \sum_{i=1}^{n} (B^*A)_{ii}$.

Moral exc (i) $m=1$ recovers std inner prod on $\mathbb{R}^n$, $\mathbb{C}^n$
(ii) pos def

Prop For $(V, \langle , \rangle)$ an inner product space over F, $\forall x, y, z \in V$, $\lambda \in F$,

$$\left.\begin{array}{l} \text{1) } \langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle \\ \text{2) } \langle x, \lambda y\rangle = \overline{\lambda}\langle x, y\rangle \\ \text{3) } \langle x, 0\rangle = \langle 0, y\rangle = 0 \end{array}\right\} \text{sesquilinearity}$$

1.5

4) If $\langle x, y\rangle = \langle x, z\rangle$ $\forall x \in V$, then $y = z$.

Pf For 1), $\langle x, y+z\rangle = \overline{\langle y+z, x\rangle}$
$= \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle}$

$= \langle x, y \rangle + \langle x, z \rangle$.

② ③ Moral exc's.

④ If $\langle x,y \rangle = \langle x,z \rangle$ $\forall x$, then

$$
\begin{aligned}
0 &= \langle x,y \rangle - \langle x,z \rangle \\
&= \langle x,y \rangle + (-1)\langle x,z \rangle \\
&= \langle x,y \rangle + \overline{(-1)}\langle x,z \rangle \\
&= \langle x,y \rangle + \langle x,(-1)z \rangle \\
&= \langle x,y \rangle + \langle x,-z \rangle \\
&= \langle x, y-z \rangle \quad \text{for all } x.
\end{aligned}
$$

In particular, for $x = y-z$, we get $0 = \langle y-z, y-z \rangle$

$\Rightarrow y-z=0$ by pos def. $\square$

<u>Defn</u> Let $(V, \langle , \rangle)$ be an IPS over $F = \mathbb{R}$ or $\mathbb{C}$. The **norm** or **length** of $x \in V$ is $\|x\| := \sqrt{\langle x,x \rangle} \in \mathbb{R}$. Two vectors $v, w \in V$ are **orthogonal** or **perpendicular** when $\langle x,y \rangle = 0$. A **unit vector** is a vector of norm 1: $\|x\| = 1 \Leftrightarrow \langle x,x \rangle = 1$.

<u>E.g.</u> (a) $V = \mathbb{R}^n$, $\langle x,y \rangle = x \cdot y$ then

$$\|x\| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

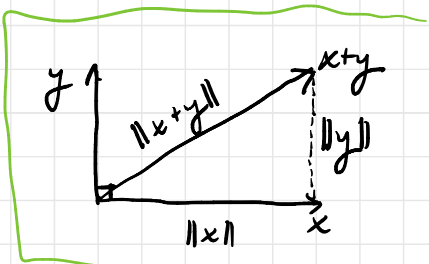
(b) $V = \mathbb{C}^n$, $\langle x, y\rangle = x \cdot \bar{y}$ then for $z \in \mathbb{C}^n$,

$$\|z\| = \sqrt{z \cdot \bar{z}} = \sqrt{\sum_{i=1}^{n} z_i \bar{z}_i} = \sqrt{\sum_{i=1}^{n} |z_i|^2}.$$

If we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via $x + iy \mapsto (x, y)$ then this matches the norm on $\mathbb{R}^n$.

Thm (Pythagoras redux) Let $(V, \langle\,,\rangle)$ be an IPS over $F = \mathbb{R}$ or $\mathbb{C}$ and let $x, y \in V$ be orthogonal. Then

$$\|x\|^2 + \|y\|^2 = \|x+y\|^2$$



Pf Know $\langle x, y\rangle = 0$, so $\langle y, x\rangle = \overline{\langle x, y\rangle} = 0$ as well.

Therefore $\|x+y\|^2 = \langle x+y, x+y\rangle$

$$= \langle x, x\rangle + \cancelto{0}{\langle x, y\rangle} + \cancelto{0}{\langle y, x\rangle} + \langle y, y\rangle$$

$$= \|x\|^2 + \|y\|^2.$$