Day 27

Learning Goals

- Powers of diagonalizable matrices
- Graphs via matrices
- Matrix powers and counting walks on graphs

Suppose $A \in \mathrm{Mat}_{n\times n}(F)$ is diagonalizable. Then $A = P^{-1}DP$ for $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and $P \in GL_n(F)$.

What are the powers of A?

$$\begin{aligned} A^2 &= (P^{-1}DP)(P^{-1}DP) \\ &= P^{-1}D(PP^{-1})DP \\ &= P^{-1}D^2P \\ &= P^{-1}\mathrm{diag}(\lambda_1^2, \dots, \lambda_n^2)P. \end{aligned}$$

In gen'l, $D^k = \mathrm{diag}(\lambda_1^k, \dots, \lambda_n^k)$ and

$$\begin{aligned} A^k &= P^{-1}DP\,A^{k-1} \\ &= P^{-1}DPP^{-1}D^{k-1}P \\ &= P^{-1}DD^{k-1}P \\ &= P^{-1}D^kP \end{aligned}$$

$$\boxed{A^k = P^{-1}\operatorname{diag}(\lambda_1^k, \dots, \lambda_n^k)\, P}$$

by induction. This is much easier than computing A^k via $k-1$ $n \times n$ matrix multiplications!

Graphs via matrices

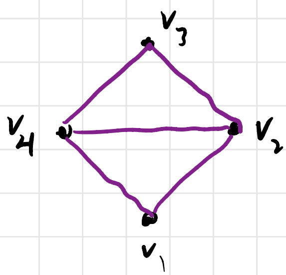
A (simple) graph is $G = (V, E)$.

$$G = (\underbrace{V}_{\text{vertices}}, \underbrace{E}_{\text{edges} \subseteq \binom{V}{2}})$$

E.g. $V = \{v_1, v_2, v_3, v_4\}$

$E = \{\{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_4\}, \{v_2, v_3\}, \{v_3, v_4\}\}$

Better to visualize as



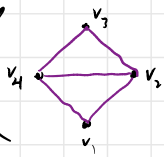
A walk (of length ℓ) in G is a sequence of vertices $u_0 u_1 \cdots u_\ell$ such that $\{u_i, u_{i+1}\} \in E(G)$ for $i = 0, \dots, \ell-1$.

In the above example, $v_1 v_4$ and $v_1 v_2 v_3 v_4$ are walks from v_1 to v_4 of length 1 and 3, resp.

Defn Let $G=(V,E)$ be a graph with $V=\{v_1,\dots,v_n\}$. The *adjacency matrix* of G is the $n\times n$ matrix $A=A(G)$ with

$$A_{ij}=\begin{cases}1 & \{v_i,v_j\}\in E\\ 0 & \text{o/w}\end{cases}.$$

For instance

$$A\left(\vphantom{\begin{matrix}0\\0\\0\\0\end{matrix}}\right.$$


$$\left.\vphantom{\begin{matrix}0\\0\\0\\0\end{matrix}}\right)=\begin{pmatrix}0&1&0&1\\1&0&1&1\\0&1&0&1\\1&1&1&0\end{pmatrix}.$$

Thm If $A=A(G)$, then the number of walks from v_i to v_j of length l in G is $\underbrace{(A^l)_{ij}}_{ij\text{-entry of } A^l}$.

Pf HW! $\square$

E.g. For G the above diamond graph, we have

$$A^0=I_4,\quad A=\begin{pmatrix}0&1&0&1\\1&0&1&1\\0&1&0&1\\1&1&1&0\end{pmatrix},\quad A^2=\begin{pmatrix}2&1&2&1\\1&3&1&2\\2&1&2&1\\1&2&1&3\end{pmatrix},\quad A^3=\begin{pmatrix}2&5&2&5\\5&4&5&5\\2&5&2&5\\5&5&5&4\end{pmatrix}.$$

The highlighted entries indicate there is 1 length 2 walks from v_2 to v_3 and 4 length 3 walks from v_2 to v_2.

TPS Find these walks!

So we'd like to compute powers of adjacency matrices efficiently.

Defn A matrix A is symmetric when $A = A^T$.

Note Adjacency matrices are symmetric.

Thm [Cor of the Spectral Thm] If A is an $n \times n$ symmetric matrix with real entries, then A is diagonalizable. $\square$

So adjacency matrices are diagonalizable:

$$A = P^{-1} D P \quad , \quad D = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

and we can compute A^ℓ as

$$A^\ell = P^{-1} D^\ell P = P^{-1} \text{diag}(\lambda_1^\ell, \ldots, \lambda_n^\ell) P.$$

As such, for each i, j $\exists$ $c_1, \ldots, c_n \in \mathbb{R}$ (independent of ℓ) such that the # of length ℓ walks in G

from v_i to v_j is $\sum_{i=1}^{n} c_i \lambda_i^l$. (!)

Defn A walk is *closed* if it starts and ends at the same vertex.

Cor The number of length l closed walks in G is $\mathrm{tr}(A^l)$. $\square$

(Recall $\mathrm{tr}(B) = \sum_{i=1}^{n} B_{ii}$ is the *trace* of B.)

Prop For $A \in \mathrm{Mat}_{n \times n}(F)$, $\mathrm{tr}(A)$ = sum of eigenvalues of A counted according to algebraic multiplicity.

Note Every field F has an *algebraic closure* $\overline{F} \supseteq F$ where $\overline{F}$ is an algebraically closed field. By viewing $\chi_A(x) \in \overline{F}[x]$ we have $\chi_A(x) = c(x-\lambda_1)\cdots(x-\lambda_n)$ for some $\lambda_i \in \overline{F}$. We are saying that $\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i$. In particular, this sum is in F, even when the λ_i are not!

Pf of Prop $\exists P \in GL_n(\overline{F})$ s.t. $P^{-1}AP = J$ is in Jordan form. The eigenvalues of A appear along the diagonal of J, alg multiplicity-many times.

We observed previously that $\mathrm{tr}(UV) = \mathrm{tr}(VU)$ for all matrices U, V, so

$$
\begin{aligned}
\mathrm{tr}(A) &= \mathrm{tr}(PJP^{-1}) \\
&= \mathrm{tr}(PP^{-1}J) \\
&= \mathrm{tr}(J) \\
&= \text{sum of eigenvalues (with alg mult).}
\end{aligned}
$$

$\square$

Cor Suppose $A = A(G) \in \mathrm{Mat}_{n \times n}(\mathbb{R})$ with eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ listed with alg multiplicity. Then the number of length l closed walks in G is

$$\lambda_1^l + \lambda_2^l + \dots + \lambda_n^l.$$

$\square$

E.g. For the diamond graph G,

$$\chi_A(x) = x(x+1)(x^2 - x - 4)$$

with roots (eigenvalues)

$$0, -1, \frac{1+\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}.$$

Thus the # closed walks in G of length ℓ is

$$w(\ell) = 0^\ell + (-1)^\ell + \left(\frac{1+\sqrt{17}}{2}\right)^\ell + \left(\frac{1-\sqrt{17}}{2}\right)^\ell$$

where $0^\ell = \begin{cases} 1 & \ell = 0 \\ 0 & \ell > 0 \end{cases}$.

Here are the first few values:

ℓ	0	1	2	3	4	5	6
$w(\ell)$	4	0	10	12	50	100	298