Day 26

Learning Goals

- Characteristic polynomial of a lin trans
- Eigenvectors with distinct eigenvalues are lin ind
- Algebraic and geometric multiplicity
- Jordan form.

(Should have noted the following earlier.)

Prop Let A, B be matrices representing a lin trans $f: V \to V$ wrt bases α, β resp. Then $\chi_A(x) = \chi_B(x)$.

Defn The characteristic polynomial of a lin trans $f: V \to V$ is $\chi_f(x) := \chi_A(x)$ for any $A = A_\alpha^\alpha(f)$.

Pf Prop We have $A = P^{-1}BP$ for some P. Thus

$$\begin{aligned}
\chi_A(x) &= \det(A - xI) \\
&= \det(P^{-1}BP - xP^{-1}IP) \\
&= \det(P^{-1}(B - xI)P) \\
&= \det(P^{-1}) \det(B - xI) \det(P)
\end{aligned}$$

Since $\det(P^{-1}) = \frac{1}{\det(P)}$ (by multiplicativity of det) we get

$$\chi_A(x) = \chi_B(x). \quad \square$$

We now check that the last bit of the diagonalization algorithm (take union of eigenspace bases) is valid.

Prop Suppose $f: V \to V$ linear with eigenvectors $v_1, \dots, v_k \in V$ corresponding to eigenvalues $\lambda_1, \dots, \lambda_k$ distinct. Then $v_1, \dots, v_k$ are lin ind.

Pf Proceed by induction on k. The $k=1$ case works b/c eigenvectors are nonzero. For induction, suppose $v_1, \dots, v_{k-1}$ lin ind for some $k>1$ and that

$$\mu_1 v_1 + \dots + \mu_k v_k = 0.$$

Apply $f - \lambda_k \mathrm{id}_V$ to this reln:

$$(f - \lambda_k \mathrm{id}_V)(\mu_1 v_1 + \dots + \mu_k v_k) = (f - \lambda_k \mathrm{id}_V)(0)$$

$$\Rightarrow \quad f(\mu_1 v_1 + \dots + \mu_k v_k) - \lambda_k(\mu_1 v_1 + \dots + \mu_k v_k)$$

$$= f(0) - \lambda_k \cdot 0$$

$$\Rightarrow \mu_1\lambda_1 v_1 + \cdots + \mu_k\lambda_k v_k - (\mu_1\lambda_k v_1 + \cdots + \mu_k\lambda_k v_k) = 0$$

$$\Rightarrow \mu_1(\lambda_1-\lambda_k)v_1 + \mu_2(\lambda_2-\lambda_k)v_2 + \cdots + \mu_k\underbrace{(\lambda_k-\lambda_k)}_{0}v_k = 0$$

$$\Rightarrow \mu_1(\lambda_1-\lambda_k)v_1 + \cdots + \mu_{k-1}(\lambda_{k-1}-\lambda_k)v_{k-1} = 0$$

$$\Rightarrow \mu_1(\lambda_1-\lambda_k) = \cdots = \mu_{k-1}(\lambda_{k-1}-\lambda_k) = 0$$ (b/c $v_1,\ldots,v_{k-1}$ lin ind)

$$\Rightarrow \mu_1 = \cdots = \mu_{k-1} = 0$$ (b/c λ_i distinct)

$$\Rightarrow \mu_k v_k = 0 \Rightarrow \mu_k = 0 \text{ too.}$$

Thus $v_1,\ldots,v_k$ lin ind, completing the induction. $\square$

Cor If $f: V \to V$ has $\dim V$ distinct eigenvalues, then f is diagonalizable.

Converse is not true in gen'l: can have repeated eigenvalues and be diag'le or non-diag'le.

Now turn to alg & geom multiplicity.

Defn A polynomial $p(x) \in F[x]$ splits over F when $\exists\, c, \lambda_1, \ldots, \lambda_n \in F$ s.t. $p(x) = c(x-\lambda_1)\cdots(x-\lambda_n)$.

E.g. x^2+1 splits over $\mathbb{C}$, but not over $\mathbb{R}$.

Fundamental Thm of Algebra Every $p(x) \in \mathbb{C}[x]$ splits over $\mathbb{C}$.

Prop Suppose $f: V \to V$ linear, $\dim V = n < \infty$. Then f diag'le $\Rightarrow$ $\chi_f(x)$ splits over F.

Pf If f diag'le then there is a basis α of V s.t. $A_\alpha^\alpha(f) = D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in F$.

Then $\chi_f(x) = \chi_D(x) = (\lambda_1 - x)\cdots(\lambda_n - x)$

$$= (-1)^n (x-\lambda_1)\cdots(x-\lambda_n)$$

which splits over F. $\square$

Converse fails: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with $\chi = (x-1)^2$ but $\dim E_1 = 1$.

Defn Let $\dim V < \infty$. The *geometric multiplicity* of an eigenvalue λ of a lin trans $f: V \to V$ is

$$\dim E_\lambda(f).$$

The *algebraic multiplicity* of λ is the number of times $(x-\lambda)$ divides $\chi_f(x)$.

Prop The geometric mult of an eigenvalue of f is $\leq$ alg mult of λ.

Pf Let $v_1, \dots, v_k$ be a basis of $E_\lambda(f)$; extend to basis $v_1, \dots, v_n$ of V. The matrix for f wrt this basis looks like $A := \begin{pmatrix} \lambda I_k & B \\ 0 & C \end{pmatrix}$.

The char poly of f is thus

$$\chi_f(x) = \det \begin{pmatrix} (\lambda - x) I_k & B \\ 0 & C - x I_{n-k} \end{pmatrix}$$

$$= (\lambda - x)^k \chi_C(x)$$

$\Rightarrow$ k = geom mult of $\lambda \leq$ alg mult of λ. $\square$

Preview of Jordan form

A **Jordan block** for λ of size k is a $k \times k$ matrix of the form $\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} =: J_k(\lambda)$.

E.g. $J_4(3) = \begin{pmatrix} 3 & 1 & & \\ & 3 & 1 & \\ & & 3 & 1 \\ & & & 3 \end{pmatrix}$

$$J_1(\lambda) = (\lambda)$$

A matrix is in Jordan form when it is a block sum of Jordan blocks:

$$\begin{pmatrix} J_{k_1}(\lambda_1) & & & 0 \\ & J_{k_2}(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J_{k_r}(\lambda_r) \end{pmatrix}.$$

E.g.
$$\begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & & 0 \\ & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & \\ & & \boxed{5} & \\ 0 & & & \boxed{\begin{matrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{matrix}} \end{pmatrix} = J_2(2) \oplus J_2(2) \oplus J_1(5) \oplus J_3(4).$$

Thm Suppose $\dim V < \infty$ and $f: V \to V$ linear with $\chi_f(x)$ splitting over F. Then $\exists$ ord basis of V s.t. matrix for f wrt this basis is in

Jordan form. The Jordan form of f is unique up to permutation of Jordan blocks.

Pf "Requires" the structure theorem for modules over a principal ideal domain! Cf. Math 332. $\square$

Note • f is diag'le iff all its Jordan blocks have size 1.

• A matrix like $\begin{pmatrix} 3 & 1 & \\ & 3 & \\ & & 2 \end{pmatrix} = J_2(3) \oplus J_1(2)$

is not diag'le.