Day 25

Learning Goals

- Diagonalizability
- Eigenspaces
- Diagonalization algorithm

Recall that $f : V \longrightarrow V$ has eigenvector v with eigenvalue λ when $v \neq 0$ and $f(v) = \lambda v$.

Defn For $\dim V = n$, a linear transformation $f : V \to V$ is *diagonalizable* when $\exists$ ordered basis $\alpha = \langle v_1, \dots, v_n \rangle$ of V s.t. $A_\alpha^\alpha(f) = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ for some $\lambda_i \in F$. A matrix $A \in \mathrm{Mat}_{n \times n}(F)$ is *diagonalizable* when $f_A : F^n \longrightarrow F^n$ is diagonalizable.

Prop A linear transformation is diagonalizable iff it has a basis of eigenvectors.

Pf We have $A_\alpha^\alpha(f) = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

$$\iff f(v_i) = \lambda_i v_i, \quad i = 1, \dots, n.$$

E.g. $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ is <u>not</u> diagonalizable unless $\theta = n\pi$, $n \in \mathbb{Z}$.

Recall that $\alpha = \langle v_1, \dots, v_n \rangle$ a basis of eigenvectors for $A \in \mathrm{Mat}_{n\times n}(F)$ and $P = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$ implies $D = P^{-1}AP$ is diagonal w/ eigenvalues on diagonal.

column vectors

Defn Matrices $A, B \in \mathrm{Mat}_{n\times n}(F)$ are *conjugate* when $\exists\, P \in \mathrm{Mat}_{n\times n}(F)$ invertible with $A = P^{-1}BP$.

TPS Conjugacy is an equivalence relation!

Rmk A, B are conjugate iff $\exists\, f: F^n \to F^n$ linear and ordered bases α, β of F^n s.t.

$$A = A_\alpha^\alpha(f), \quad B = A_\beta^\beta(F).$$

Recall that $\chi_A(x) = \det(A - xI_n)$ is the characteristic polynomial of A and that $\lambda \in F$ is an eigenvalue of A iff $\chi_A(\lambda) = 0$.

E.g. If $A = \begin{pmatrix} 2 & -7 & 3 \\ 0 & -5 & 3 \\ 0 & 0 & 2 \end{pmatrix}$ then

upper triangular!

$$\chi_A(x) = \det\begin{pmatrix} 2-x & -7 & 3 \\ 0 & -5-x & 3 \\ 0 & 0 & 2-x \end{pmatrix} = -(x-2)^2(x+5).$$

Thus A has eigenvalues 2 (with multiplicity 2) and -5.

Defn Let λ be an eigenvalue of a matrix A. The *eigenspace* of A for λ is

$$E_\lambda(A) := \{v \in V \mid Av = \lambda v\} = \ker(A - \lambda I_n).$$

To (attempt to) diagonalize A, we must find bases for all of its eigenspaces.

E.g. (ct'd) To compute E_2:

$$A - 2I_3 = \begin{pmatrix} 2 & -7 & 3 \\ 0 & -5 & 3 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -7 & 3 \\ 0 & -7 & 3 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & -3/7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow E_2 = \ker(A - 2I_3) = \{(x, \tfrac{3}{7}z, z) \mid x, z \in \mathbb{R}\}$$

with basis $\langle (1,0,0), (0, \tfrac{3}{7}, 1) \rangle$

or $\langle (1,0,0), (0,3,7) \rangle$.

To compute E_{-5}:

$$A - 5I_3 = \begin{pmatrix} 7 & -7 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow E_{-5} = \ker(A - 5I_3) = \{(y, y, 0) \mid y \in \mathbb{R}\}$$

with basis $\langle (1,1,0) \rangle$.

We will see later that eigenvectors in distinct eigenspaces are linearly ind, so $\langle (1,0,0), (0,3,7), (1,1,0) \rangle$ is a basis of eigenvectors for A.

Hence $\operatorname{diag}(2, 2, -5) = P^{-1} A P$

for $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 7 & 0 \end{pmatrix}$.

E.g. Let's modify A slightly:

$$A' = \begin{pmatrix} 2 & 1 & 3 \\ 0 & -5 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

This has the same char poly and same eigenvalues as A.

A basis for $E_{-5}(A')$ is $\langle (-7, 1, 0) \rangle$. (Similar steps as before.)

TPS Find a basis for $E_2(A')$.

Should get $A-2I_3 \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow E_2(A')$ has basis $\langle (1,0,0) \rangle$.

Thus there are at most two lin ind eigenvectors for A'! We conclude that A' does not admit a basis of eigenvectors $\Rightarrow A'$ is not diagonalizable.

Diagonalization Algorithm

(1) Find eigenvalues of A as roots of χ_A.

(2) For eigenvalue λ of A, compute a basis of $E_\lambda(A)$.

(3) The matrix A is diagonalizable iff the total number of basis vectors found in (2) is n.
If so, these vectors form an eigenbasis for A and if P is the matrix with these vectors as columns, then

$$D = P^{-1}AP$$

is diagonal w/ eigenvalues on its diagonal.

Note We still need to show that vectors from different eigenspaces are lin ind. Next time!

Defn If λ is an eigenvalue of A, its (algebraic) multiplicity is the # of factors of $(x-\lambda)$ in $\chi_A(x)$.

The geometric multiplicity of λ is $\dim E_\lambda(A)$.

We always have $\sum$ geom mult's $\leq \sum$ alg mult's and A is diagonalizable iff <u>both</u> of these sums are n iff $\sum$ geom mult's $= n$.