Day 24

Learning Goals

- Define eigenvectors + eigenvalues of linear transformations.
- Eigenvalues are roots of the characteristic polynomial

Defn Let $f: V \to V$ be a linear transformation. A nonzero vector $v \in V$ is an **eigenvector** of f with **eigenvalue** $\lambda \in F$ when

$$f(v) = \lambda v.$$

What's good about eigenvectors? Suppose $\alpha = \langle v_1, \dots, v_n \rangle$ is an ordered basis of eigenvectors for f with eigenvalues $\lambda_1, \dots, \lambda_n$. Then

$$A_\alpha^\alpha(f) = \mathrm{diag}(\lambda_1, \dots, \lambda_n).$$

$$= \begin{pmatrix} \lambda_1 & & \bigcirc \\ & \lambda_2 & \\ \bigcirc & \ddots & \lambda_n \end{pmatrix}.$$

Eigenvector bases don't always exist!

Diagonal matrices are much easier to manipulate.

TPS Compute the powers of $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$.

Does this generalize to larger diagonal matrices?

E.g. Let $A = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}$ with associated linear trans $f_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Since

$$\begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

and $\begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

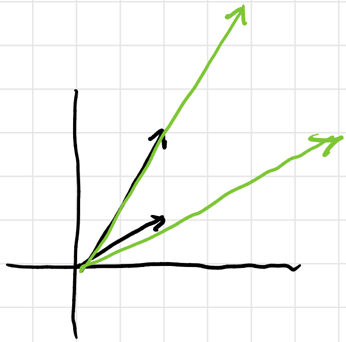
we know that $(2,3)$ and $(1,2)$ are eigenvectors of f_A (we could – and will – call them eigenvectors of A) with eigenvalues $2, 3$ respectively.

Let $\alpha = \langle (2,3), (1,2) \rangle$. Since

$$f_A(2,3) = 2 \cdot (2,3) + 0 \cdot (1,2)$$

$$f_A(1,2) = 0 \cdot (2,3) + 3 \cdot (1,2),$$

we have $A_\alpha^\alpha(f_A) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} =: D$.



In terms of change of basis:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] \arrow[d, "\mathrm{Rep}_\alpha"'] & \mathbb{R}^2 \arrow[d, "\mathrm{Rep}_\alpha"] \\ \mathbb{R}^2 \arrow[r, "D"'] & \mathbb{R}^2 \end{tikzcd}$$

$$(1,2) \mapsto 3\cdot(1,2) \qquad (2,3) \mapsto 2\cdot(2,3)$$

$$(1,0) \mapsto 2\cdot(1,0) \qquad (0,1) \mapsto 3(0,1)$$

The matrix for Rep_α^{-1} is $P = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ and the matrix for Rep_α is P^{-1}. Thus

$$D = P^{-1}AP$$

Summary Eigenvectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are columns of P. Then $D = P^{-1}AP$ is a diagonal matrix with eigenvalues on the diagonal.

More generally Suppose $f: F^n \longrightarrow F^n$ is a lin trans and $\alpha = \langle v_1, \dots, v_n \rangle$ is a basis of eigenvectors for f of V with eigenvalues $\lambda_1, \dots, \lambda_n$. Let P have columns $v_1, \dots, v_n$ and let A be the matrix inducing f (with cols $f(e_1), \dots, f(e_n)$). Then

$$D = P^{-1}AP = \mathrm{diag}(\lambda_1, \dots, \lambda_n).$$

How to find eigenvectors and eigenvalues

For $A \in \mathrm{Mat}_{n \times n}(F)$, $v \in F^n$, $\lambda \in F$,

$$Av = \lambda v \iff (A - \lambda I_n) v = 0$$

$$\iff v \in \ker(A - \lambda I_n)$$

Thus λ is an eigenvalue of A iff $\ker(A - \lambda I) \neq \{0\}$.

But $\ker(A - \lambda I) \neq \{0\} \iff \mathrm{rank}(A - \lambda I) < n$

$$\iff \det(A - \lambda I) = 0.$$

Lets work this out in our example:

$$\det\left(\begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det \begin{pmatrix} -1-\lambda & 2 \\ -6 & 6-\lambda \end{pmatrix}$$

$$= (-1-\lambda)(6-\lambda) + 12$$

$$= \lambda^2 - 5\lambda + 6$$

$$= (\lambda - 2)(\lambda - 3).$$

Thus the eigenvalues are $\lambda = 2, 3$.

How do we find the eigenvectors?

Need to compute $\ker(A-\lambda I_n)$ for eigenvalues λ.

In the above case,

$$A-2I_2 = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 2 \\ -6 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2/3 \\ 0 & 0 \end{pmatrix}$$

so $\ker(A-2I_2) = \{(\frac{2}{3}y, y) \mid y \in \mathbb{R}\}$

$= \operatorname{span}\{(\frac{2}{3}, 1)\}$.

Similarly,

$$A-3I_2 = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}$$

so $\ker(A-3I_2) = \operatorname{span}\{(\frac{1}{2}, 1)\}$.

TPS Find a matrix that is not diagonalizable.
(Think geometrically?)