Day 23

Learning Goals

- Laplace expansion of det
- Existence and uniqueness of det

For $A \in \mathrm{Mat}_{n\times n}(F)$ and $1 \le i,j \le n$, define $A^{ij} \in \mathrm{Mat}_{(n-1)\times(n-1)}(F)$ to be the matrix formed from A by deleting its i-th row and j-th column.

E.g. If $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$ then

$$A^{23} = \begin{pmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{pmatrix}.$$

The ij in A^{ij} is an index, not a power!

Thm [Laplace expansion] Suppose $A \in \mathrm{Mat}_{n\times n}(F)$ and fix $k \in \{1, \dots, n\}$. Then

$$\det A = \sum_{j=1}^{n} (-1)^{k+j} A_{kj} \det(A^{kj})$$

We call this the expansion of det A along the k-th row of A.

E.g.

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (-1)^{1+1} a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

$$+ (-1)^{1+2} a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + (-1)^{1+3} a_{13} \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$
$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Note $(-1)^{k+j}$ makes a "sign checkerboard":

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This creates a helpful mnemonic when expanding along an arbitrary row.

TBS Use Laplace expansion to determine

$$\det\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 0 \\ 1 & 4 & 1 \end{pmatrix} \qquad \begin{matrix} + & - & + \\ \boxed{- \quad + \quad -} \\ + & - & + \end{matrix}$$

Ans Easiest to expand along second row:

$$-0\cdot(\sim) + 2\,(1\cdot 1 - 2\cdot 1) - 0\cdot(\sim)$$

$$= 2(-1) = -2.$$

Cor [Laplace expansion along columns]

Since $\det A = \det A^T$, we can also compute $\det A$ by expanding along columns:

For $A \in \mathrm{Mat}_{n\times n}(F)$ and $1 \le k \le n$ fixed,

$$\det A = \sum_{i=1}^{n} (-1)^{i+k} A_{ik} \det(A^{ik}).$$

E.g. Expanding along first column,

$$\det\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 0 \\ 1 & 4 & 1 \end{pmatrix} = 1\cdot(2\cdot 1 - 0\cdot 4) - 0\cdot(3\cdot 1 - 2\cdot 4) + 1\cdot(3\cdot 0 - 2\cdot 2)$$

$$= 2 - 4 = -2. \quad \checkmark$$

Pf Sketch for Laplace Expansion Thm

By permutation expansion,

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \cdot \prod_{i=1}^{n} A_{i\sigma(i)}.$$

Factor out all the A_{kj} terms:

$$\det A = \sum_{j=1}^{n} A_{kj} \underbrace{\sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(k)=j}} \operatorname{sgn}(\sigma) \prod_{\substack{1 \le i \le n \\ i \ne k}} A_{i\sigma(i)}}_{\text{Need to show this is } (-1)^{k+j} \det A^{kj}.}$$

The ~ term has all the correct A_{ij} terms and (moral exercise) you can check that the sign is off by $(-1)^{k+j}$.

(by swapping to get k-th rows in place)

See Hefferon for 3×3 case.

□

Thm A determinant function (multilinear, alternating in rows, normalized) exists and is unique.

Pf Sketch Make the following recursive def'n of a function $d: \mathrm{Mat}_{n\times n}(F) \to F$:

$$\underline{n=1} \quad d(a) = a$$

$$\underline{n>1} \quad d(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j}\, d(A^{1j})$$

(inspired by laplace exp'n along first row)

Do some tedious work to check that d is multilinear, alternating, normalized.

Thus d is <u>a</u> determinant function.

Our work with det + row ops shows that <u>all</u> determinant functions are determined by a sequence of row ops to REF. Thus any two det fns take the same values! $\square$