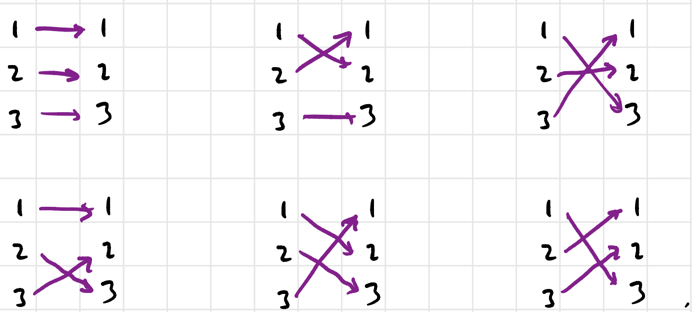
Day 22

Learning Goals

- Permutation matrices
- Sign of a permutation
- Permutation expansion of det.
- Vandermonde det

Defn A permutation of a set X is a bijective function $\sigma : X \longrightarrow X$. If τ is another permutation of X, then so is $\sigma \circ \tau$. The set $\mathfrak{S}_X$ of permutations of X together with the binary operation $\circ$ is called the symmetric group of X. If $X = \{1, 2, \ldots, n\}$, then we denote this by $\mathfrak{S}_n$.

E.g. Here are the six elements of $\mathfrak{S}_3$:



In general, there are $n! = n(n-1)\cdots 2\cdot 1$ elements of $\mathfrak{S}_n$: n choices for $\sigma(1)$

$n-1$ choices for $\sigma(2) \neq \sigma(1)$

$n-2$ choices for $\sigma(3) \neq \sigma(1), \sigma(2)$ etc.

Defn For $\sigma \in \mathfrak{S}_n$, the **permutation matrix** corresponding to σ is $P_\sigma \in \mathrm{Mat}_{n\times n}(F)$ with i-th row $e_{\sigma(i)}$.
In other words, P_σ is obtained from I_n by permuting its columns according to σ.

E.g. Suppose
$$\sigma = \begin{tikzcd}[row sep=small] 1 \arrow[rd] & 1 \\ 2 \arrow[rd] & 2 \\ 3 \arrow[ruu] & 3 \end{tikzcd}.$$
Then
$$P_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Moral Exercises

(1) If the rows of A are $r_1, \ldots, r_n$, then $P_\sigma A$ has i-th row $r_{\sigma(i)}$.

(2) $P_\sigma e_{\sigma(i)} = e_i$

(3) $P_\sigma P_\tau = P_{\tau\circ\sigma}$ ⚠ Order of σ, τ switches!

Spoiler for Math 113 Perm'n matrices correspond to non-attacking rook placements.

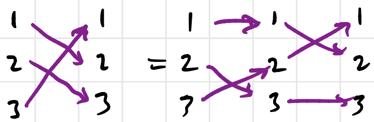
Defn The **sign** of a permutation $\sigma \in \mathfrak{S}_n$ is
$$\mathrm{sgn}(\sigma) := \det(P_\sigma) = \pm 1.$$
A perm'n is **even** if its sign is 1, and **odd** if its sign

is -1.

swaps

Fact Every perm'n is a composition of transpositions, so P_σ is obtained from I_n by some number of column swaps. Each swap changes the value of det by multiplying by -1 (and $\det I_n = 1$), so $\operatorname{sgn}(\sigma) \in \{\pm 1\}$.

E.g.



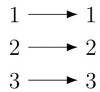
is even.

Note Every permutation can be expressed in many ways as a composition of transpositions. Well-def'n of det implies that the number of transpositions in such a decomposition is always even ($\operatorname{sgn}(\sigma) = 1$) or else is always odd ($\operatorname{sgn}(\sigma) = -1$).

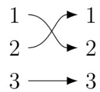
Thm For $A \in \mathrm{Mat}_{n \times n}(F)$,

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}.$$

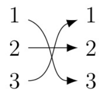
E.g. Adding up the following six terms gives the determinant of a 3×3 matrix:



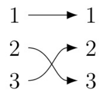
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad a_{11}a_{22}a_{33}$$



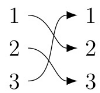
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad -a_{12}a_{21}a_{33}$$



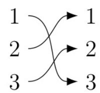
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad -a_{13}a_{22}a_{31}$$



$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad -a_{11}a_{23}a_{32}$$



$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad a_{12}a_{23}a_{31}$$



$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad a_{13}a_{21}a_{32}$$

TPS Check that this works for 2×2 matrices.

Pf of Thm We want to compute

$$\det A = \det(A_{11}e_1 + A_{12}e_2 + \cdots + A_{1n}e_n, \ldots, A_{n1}e_1 + A_{n2}e_2 + \cdots + A_{nn}e_n)$$

Expanding by multilinearity, we get n^n terms that look like

$$A_{1j_1} A_{2j_2} \cdots A_{nj_n} \det(e_{j_1}, e_{j_2}, \ldots, e_{j_n}).$$

But if $j_k = j_\ell$ for any $k \neq \ell$, then e_{j_k} will be duplicated in the determinant expression which will thus be 0 by the alternating property of det.

As such, the only terms contributing (potentially) nonzero terms to $\det A$ are of the form

$$A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)} \det(\underbrace{e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(n)}}_{\text{rows of } P_\sigma})$$

$$= \operatorname{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}$$

for $\sigma \in \mathfrak{S}_n$. $\square$

Vandermondia

Suppose we have n points $(x_1, y_1), \dots, (x_n, y_n) \in \mathbb{R}^2$.

Given that 2 points determine a line, 3 points determine a parabola, etc, we expect that there is a degree $n-1$ polynomial $p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$ *interpolating* between the points: $p(x_i) = y_i$, $1 \le i \le n$.

Each such constraint corresponds to the linear equation $a_0 + a_1 x_i + a_2 x_i^2 + \cdots + a_{n-1} x_i^{n-1} = y_i$.

To determine $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$, we turn to the augmented matrix

$$\underbrace{\left(\begin{array}{ccccc|c} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & y_1 \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & y_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & y_n \end{array}\right)}_{\text{Vandermonde matrix } V}.$$

The system has a solution when $\det V \neq 0$.

Consider the linear transformation

$$f: \mathbb{R}[x]_{\le n-1} \longrightarrow \mathbb{R}^n$$
$$p(x) \longmapsto (p(x_1), \ldots, p(x_n)).$$

Let $\mu = \langle 1, x, \ldots, x^{n-1} \rangle$, $\mathcal{E} = \langle e_1, \ldots, e_n \rangle$.

Then $A_\mu^{\mathcal{E}}(f) = V$.

Now consider a new ordered basis

$$\alpha = \langle 1, x-x_1, (x-x_1)(x-x_2), \ldots, (x-x_1)(x-x_2)\cdots(x-x_n) \rangle$$

of $\mathbb{R}[x]_{\le n}$. The i-th term of α is monic of degree i, | x^i + lower order terms

so $A_\alpha^\mu(id_{\mathbb{R}[x]_{\le n-1}})$ is upper triangular

with 1's on the diagonal.

Thus $\det A_\alpha^\mu(id_{\mathbb{R}[x]_{\le n-1}}) = 1$.

We have $A^{\varepsilon}_{\alpha}(f) = A^{\varepsilon}_{\mu}(f)\, A^{\mu}_{\alpha}(\mathrm{id})$

but by evaluating the α polynomials at $x_1, \dots, x_n$ we also have

$$A^{\varepsilon}_{\alpha}(f) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & 0 & \cdots & 0 \\ 1 & x_3 - x_1 & (x_3 - x_1)(x_3 - x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & x_n - x_1 & (x_n - x_1)(x_n - x_2) & \cdots & (x_n - x_1)(x_n - x_2)\cdots(x_n - x_{n-1}) \end{pmatrix}.$$

Thus $\det A^{\varepsilon}_{\alpha}(f) = \det V \cancelto{1}{\det A^{\mu}_{\alpha}(\mathrm{id})}$

$$\Rightarrow \boxed{\det V = \prod_{i<j} (x_j - x_i)}.$$

Note that $\det V = 0$ iff some $x_j = x_i$, $i = j$.

If $\det V \neq 0$, then $V^{-1}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix}$, for

$$p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$$

interpolating.

$(\det V)^2$ is the discriminant of a polynomial p with distinct roots $x_1, \dots, x_n$ — important in Galois theory.