Day 21

Learning Goals

- $\det A = \det A^T$
- det is multilinear, alternating in columns as well
- Computing det with row + col ops

Defn A square matrix $A \in \mathrm{Mat}_{n\times n}(F)$ is an **elementary matrix** when it is obtained from I_n by a single row operation.

E.g. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\in \mathrm{Mat}_{n\times n}(F)$

Prop Let E be an elementary matrix corr to a particular row op. For $A \in \mathrm{Mat}_{n\times k}(F)$, EA is the matrix produced from A by that row op. $\square$

E.g.

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 0 & -1 & 2 \\ 1 & 5 & 6 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -6 & -10 & -10 \\ 1 & 5 & 6 & 7 \end{pmatrix}$$

Note By G-J red'n, $\exists$ elementary matrices $E_1, \dots, E_\ell$ s.t. $REF(A) = E_\ell E_{\ell-1} \cdots E_2 E_1 A$.

Thm For all $A \in Mat_{n\times n}(F)$, $\det A = \det A^T$.

We can quickly check the 2x2 case:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - cb = ad - bc$$

To prove the thm in gen'l, we need some add'l facts:

Thm For all $A, B \in Mat_{n\times n}(F)$,

$$\det(AB) = \det(A)\det(B).$$

det is "multiplicative"

Pf Upcoming HW! $\square$

Prop (a) When the product is defined, $(AB)^T = B^T A^T$

(b) When A is invertible $(A^T)^{-1} = (A^{-1})^T$.

Pf (a) was in HW.

(b) $id = f \circ f^{-1} \Rightarrow id = id^* = (f^{-1})^* \circ f^*$

$\Rightarrow I = (A^{-1})^T A^T$ $\square$

Lemma Let E be an elementary matrix. Then

$$\det E = \det E^T \neq 0.$$

Pf (1) If $E \xleftarrow{r_i \leftrightarrow r_j} I_n$, then $E = E^T$ and $\det E = -1 = \det E^T$.

(2) If $E \xleftarrow{\lambda r_i \leftarrow r_i} I_n$, then $E = E^T$ and $\det E = \lambda = \det E^T$.

(3) If $E \xleftarrow{r_i + \lambda r_j \leftarrow r_i} I_n$ for $i \neq j$, then $E^T \xleftarrow{r_i' + \lambda r_j' \leftarrow r_i'} I_n$

$\left(\text{e.g., } E = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ then } E^T = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$

for r_k' = k-th row of E^T. Thus $\det E = \det E^T = 1$. $\square$

Pf that $\det A = \det A^T$ First suppose $REF(A) \neq I_n$.

Then $\operatorname{rank}(A) = \operatorname{rank}(A^T) < n$ so $\det A = 0 = \det A^T$.

Now assume $REF(A) = I_n$. Then there are elementary matrices $E_1, \dots, E_\ell$ s.t.

$$I_n = E_\ell \cdots E_1 A$$

$$\Rightarrow \quad 1 = \underline{\det(E_\ell) \cdots \det(E_1)} \det(A).$$

Also $I_n = I_n^T = (E_\ell \cdots E_1 A)^T = A^T E_1^T \cdots E_\ell^T$

$$\Rightarrow \quad 1 = \det(A^T) \underline{\det(E_1^T) \cdots \det(E_\ell^T)}$$

So $\det A^T = \det A$. $\square$

<u>Cor</u> det is multilinear, alternating in columns.

<u>E.g.</u>

$$\det\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 2 \\ 3 & 1 & 3 \end{pmatrix} = 0 \text{ b/c } 1^{st}, 3^{rd} \text{ cols equal.}$$