Day 20

Learning goals

- Determinant as function $\mathrm{Mat}_{n\times n}(F) \xrightarrow{\det} F$ which is alternating & multilinear in rows and normalized: $\det(I_n) = 1$.
- Compute det via row ops
- $\det A \neq 0 \iff \operatorname{rank} A = n \iff A$ invertible.

We are investigating a (putative) function $\det : \mathrm{Mat}_{n\times n}(F) \to F$. If A has rows $r_1, \ldots, r_n$, write

$$\det A = \det(r_1, \ldots, r_n).$$

Properties of det:

Multilinear: det is linear in each of its variables

Alternating: If $r_i = r_j$ for some $i \neq j$, then $\det(r_1, \ldots, r_n) = 0$.

Normalized: $\det(I_n) = \det(e_1, \ldots, e_n) = 1$.

Thm For each $n \geq 0$, there is a unique determinant function satisfying the above properties.

Proof — later!

For now, we assume det exists and explore consequences.

Prop [behavior of det wrt row ops]

(1) If $A \xrightarrow{r_i \leftrightarrow r_j} B$ then $\det A = -\det B$

(2) If $A \xrightarrow{r_i \to \lambda r_i} B$ then $\det B = \lambda \det A$

(3) If $A \xrightarrow{r_i \to r_i + \lambda r_j} B$ then $\det A = \det B$.

Pf Sketch

(1) $0 = \det(r_1 + r_2, r_1 + r_2)$

$$= \det(r_1, r_1) + \det(r_1, r_2) + \det(r_2, r_1) + \det(r_2, r_2)$$

$$= 0 + \det A + \det B + 0$$

$$= \det A + \det B$$

Same argument for bigger matrices, different rows.

(2) Multilinearity.

(3) $\det(r_1, \lambda r_1 + r_2)$

$$= \lambda \det(r_1, r_1) + \det(r_1, r_2)$$

$$= 0 + \det(r_1, r_2)$$

This argument also generalizes. $\square$

E.g.

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\left((a,b),\ (c,d)\right)$$

$$= a\det(e_1,\ (c,d)) + b\det(e_2,\ (c,d))$$

$$= ac\det(e_1,e_1) + ad\det(e_1,e_2)$$

$$+ bc\det(e_2,e_1) + bd\det(e_2,e_2)$$

$$= ad\cdot 1 - bc\cdot 1$$

$$= ad - bc.$$

E.g. We can use G-J red'n to compute det!

$$\det(A) = \det\begin{pmatrix} 1 & 2 & -2 \\ 9 & 4 & 0 \\ 2 & 2 & 4 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 2 & -2 \\ 0 & -14 & 18 \\ 0 & -2 & 8 \end{pmatrix}$$

$$= -\det\begin{pmatrix} 1 & 2 & -2 \\ 0 & -2 & 8 \\ 0 & -14 & 18 \end{pmatrix}$$

$$= 2\det\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \\ 0 & -14 & 18 \end{pmatrix}$$

$$= 2\det\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & -38 \end{pmatrix}$$

$$= 2(-38)\det\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 2(-38)\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 2(-38) = -76.$$

E.g.

$$\det\begin{pmatrix} 4 & 2 & -3 & 8 \\ 0 & 5 & 1 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$ ← upper triangular!

$$= 4\cdot 5\cdot 2\cdot 3 \det\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= 120 \det I_4$$

$$= 120.$$

Defn A matrix is *upper triangular* if it looks like

$$\begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & 0 & & * & * \\ & & & & * \end{pmatrix}$$

i.e., A is upper triangular when $A_{ij} = 0$ for $i > j$.

Prop The determinant of an upper triangular matrix is the product of its diagonal entries.

Pf The above method generalizes as long as no diagonal entries are 0:

$$\det\begin{pmatrix} a_1 & & & * \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix} = a_1 \cdots a_n \det\begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$= a_1 \cdots a_n \det I_n$$

$$= a_1 \cdots a_n .$$

In gen'l, $\det A = \lambda \det(REF(A))$ for some $\lambda \in F \setminus \{0\}$ by G-J red'n. If A has a 0 on its diagonal, then $REF(A)$ has a row of all 0's and — pulling a 0 scalar out — we get $\det A = 0$ as desired. $\square$

Prop For $A \in Mat_{n \times n}(F)$, TFAE:

(1) $\det A \neq 0$
(2) $\operatorname{rank} A = n$
(3) A is invertible.

Pf We have already shown $(2) \Leftrightarrow (3)$, so it suffices to check $(1) \Leftrightarrow (2)$. Since $\det A = \lambda \det REF(A)$ for some $\lambda \in F \setminus \{0\}$, we know $\det A = 0 \Leftrightarrow \det REF(A) = 0$. The rank of A is n iff $REF(A) = I_n$ iff $\det A = \lambda \neq 0$.

when $REF(A) \neq I_n$, it is upper Δ'lar w/ row of 0s, so $\det = 0$. $\square$

A tour of things to come:

(1) $\det A^T = \det A$

(2) $\det AB = \det A \cdot \det B$

(3) "Laplace expansion" of det along any row or column

(4) "Permutation expansion"

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)}$$

(5) Over $\mathbb{R}$, $\det A$ gives the "signed volume" of the parallelepiped spanned by the columns of A.

And, of course, det exists and is unique!