Day 19

Learning Goals

- Composition of linear trans = matrix mult'n
- How to change bases with matrices

Thm Suppose $A \in \mathrm{Mat}_{m\times p}(F)$, $B \in \mathrm{Mat}_{p\times n}(F)$ with corresponding linear trans'ns $f_A : F^p \to F^m$, $f_B : F^n \to F^p$. Then $f_A \circ f_B = f_{AB}$ where $f_{AB} : F^n \to F^m$ is the linear trans corresponding to the matrix product $AB \in \mathrm{Mat}_{m\times n}(F)$.

Pf For $x \in F^n$,
$$\begin{aligned}(f_A \circ f_B)(x) &= f_A(f_B(x)) \\ &= f_A(Bx) \\ &= A(Bx) \\ &= (AB)x \\ &= f_{AB}(x). \quad \square\end{aligned}$$

Q How would you invent matrix mult'n?

A? (1) Decide to encode lin trans'ns according to how they act on $e_1, \dots, e_n$.

12) Ask what the affect of composition is on $e_1, \dots, e_n$ given knowledge of basis actions.

13) Record the answer in a new matrix — the "product"!

E.g. What is the matrix for the lin trans which rotates ccw by $\pi/4$ then reflects across the x-axis?

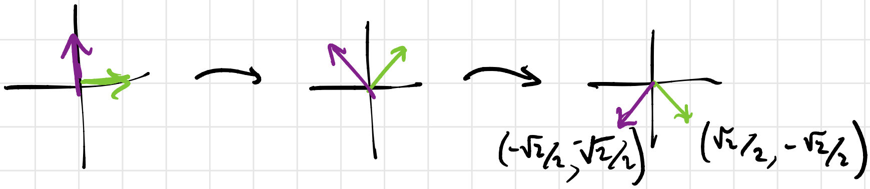
A1

$$R = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \text{ rotates by } \pi/4$$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ reflects across x-axis}$$

$$MR = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ -\sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}$$

A2



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Change of basis

Let α be an ordered basis of F^n, β an ordered basis of F^m. Suppose $f: F^n \to F^m$ is a lin trans.

We get a diagram

$$\begin{tikzcd} F^n \arrow[r, "f"] \arrow[d, "\mathrm{Rep}_\alpha"'] & F^m \arrow[d, "\mathrm{Rep}_\beta"] \\ F^n \arrow[r, "A_\alpha^\beta(f)"'] & F^m \end{tikzcd}.$$

Each map is recorded by a matrix (rel to std bases):

$$\begin{tikzcd} F^n \arrow[r, "A"] \arrow[d, "M"'] & F^m \arrow[d, "N"] \\ F^n \arrow[r, "B"'] & F^m \end{tikzcd}$$

(Here begins the typical subterfuge of conflating matrices and induced linear trans'ns.)

Suppose $\alpha = \langle v_1, \dots, v_n \rangle$, $\beta = \langle w_1, \dots, w_m \rangle$.

Let $P = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$, $Q = \begin{pmatrix} | & & | \\ w_1 & \cdots & w_m \\ | & & | \end{pmatrix}$.

Then $P = M^{-1}$, $Q = N^{-1}$ (do you see why?).

$$\begin{tikzcd} v_j & w_k \\ e_j \arrow[u] & e_k \arrow[u] \end{tikzcd}$$

Thus $B = Q^{-1} A P$.

E.g. Consider the linear trans

$$f : \mathbb{Q}^3 \longrightarrow \mathbb{Q}^2$$
$$(x, y, z) \longmapsto (x + 3y + 2z,\ 2y + z)$$

with matrix $A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \end{pmatrix}$.

Choose ordered basis

$$\alpha = \langle (1,0,0), (1,1,0), (1,1,1) \rangle$$
$$\beta = \langle (0,1), (1,1) \rangle .$$

To find $A_\alpha^\beta(f)$, create

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then $A_\alpha^\beta(f) = Q^{-1} A P$

$$= \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -2 & -3 \\ 1 & 4 & 6 \end{pmatrix}.$$

Indeed, $f(1,0,0) = (1,0) = -(0,1) + (1,1)$

$$f(1,1,0) = (4,2) = -2(0,1) + 4(1,1)$$
$$f(1,1,1) = (6,3) = -3(0,1) + 6(1,1),$$

so our computation checks!

Important special case!

$V = W = F^n$, $\alpha = \beta = \langle v_1, \dots, v_n \rangle$

Then looking at

$$\begin{tikzcd} F^n \arrow[r, "A"] \arrow[d, "P^{-1}"'] & F^n \arrow[d, "P^{-1}"] \\ F^n \arrow[r, "B"'] & F^n \end{tikzcd},$$

where $P = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$. We get

$$B = P^{-1} A P$$

and say B is formed by conjugating A by P.

E.g. Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $\alpha = \langle (1,1), (-1,0) \rangle$.

Then $P = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and we compute P^{-1} via

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[G\text{-}J]{} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}}_{P^{-1}}$$

Thus $B = P^{-1}AP$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}.$$

strange but true in this case: $B = A$. This is equivalent to P commuting with A!

We can check that

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$A\begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

verifying our comp'n. See sage worksheet for geometric view.

<u>Moral</u> <u>exc</u> Same comp'n, but with $\alpha = \langle (1,1), (1,-1) \rangle$. You should get $B = \begin{pmatrix} 3/2 & 3/2 \\ -1/2 & 3/2 \end{pmatrix}$. This $\neq A$, so $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ doesn't commute with A.

<u>Extra credit</u> Write a short story about a change of perspective that illuminates our change of basis formula.