Day 18

Learning goals

- Recall matrix - linear transformation correspondence
- $\text{im}(f_A) = \text{col space}(A) \Rightarrow \text{rank}(A) = \text{rank}(f_A)$
- Graph theory application: cycle space

We constructed a bijection

$$\begin{aligned} \text{Mat}_{m\times n}(F) &\longrightarrow \text{Hom}(F^n, F^m) \\ A &\longmapsto (f_A : x \mapsto Ax) \end{aligned}$$

as column vector

The inverse function is given by

$$\begin{pmatrix} f(e_1) & \cdots & f(e_n) \end{pmatrix} \longleftarrow f.$$

TPS Why are these inverse functions?

Answer A linear trans is determined by its action on a basis.

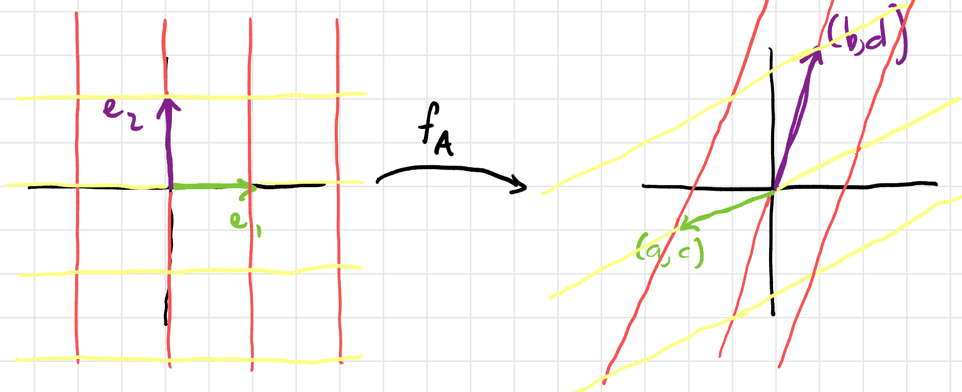
E.g. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \in \text{Mat}_{3\times 2}(F)$ induces

$$f_A : F^2 \longrightarrow F^3$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 3x+4y \\ 5x+6y \end{pmatrix}.$$

Note that $f_A(e_1) = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$, $f_A(e_2) = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$.

E.g. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{R}^2)$.

How does A act on $\mathbb{R}^2$?



TPS (1) Find A such that f_A rotates by $\pi/4$ ccw.

(2) Find A such that f_A reflects across the x-axis

(3) Find A such that f_A stretches x-axis by a factor of 3.

Recall that the image of a lin trans $f: V \to W$ is $\mathrm{im}(f) = \{f(v) \mid v \in V\}$. If $\{v_1, \dots, v_n\}$ is a basis of V, then $\mathrm{im}(f) = \mathrm{span}\{f(v_1), \dots, f(v_n)\}$ (why?).

In particular,
$$\begin{aligned} \mathrm{im}(f_A) &= \mathrm{span}\{f_A(e_1), \dots, f_A(e_n)\} \\ &= \mathrm{span}\{\text{cols of } A\} \\ &= \text{col space}(A). \end{aligned}$$

The dimension of colspace (A) is the rank of A.
The dimension of $\mathrm{im}(f_A)$ is the rank of f_A.
Thus $\mathrm{rank}(A) = \mathrm{rank}(f_A)$, so this terminology was well-chosen.

By choosing ordered bases, we can make matrices for arbitrary linear trans'ns b/w finite-dimensional vector spaces.

Suppose $f: V \to W$ linear, V has ordered basis $\alpha = \langle v_1, \dots, v_n \rangle$, W has ordered basis $\beta = \langle w_1, \dots, w_m \rangle$.

Then

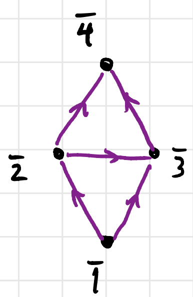
$$\begin{tikzcd} V \arrow[r, "f"] & W \arrow[d, "\mathrm{Rep}_\beta", "\cong"'] \\ F^n \arrow[u, "\mathrm{Rep}_\alpha^{-1}", "\cong"'] \arrow[r, "\mathrm{Rep}_\beta \circ f \circ \mathrm{Rep}_\alpha^{-1}"'] & F^m \end{tikzcd}$$

induces a lin trans $F^n \longrightarrow F^m$ and we call the associated matrix $A_\alpha^\beta(f) = A_\alpha^\beta \in \mathrm{Mat}_{m\times n}(F)$.

The j-th column of A_α^β is $\mathrm{Rep}_\beta f(v_j)$, i.e. the β-coordinates of the image of the j-th basis vector of V.

Graphs and matrices

Consider the following directed graph G:



$G = (V, E)$ with

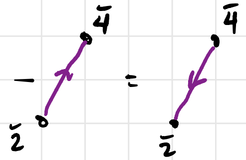
$V = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$

$E = \{\overline{12}, \overline{13}, \overline{23}, \overline{24}, \overline{34}\}$.

Define the edge space $\mathbb{Q}E$ = $\mathbb{Q}$-vector space with basis E. A typical elt of $\mathbb{Q}E$ is a formal linear combo of elts of E, such as

$$2\cdot\overline{12} - \frac{3}{2}\cdot\overline{13} + 14\cdot\overline{23} - \frac{8}{3}\cdot\overline{34}.$$

We may interpret negative signs as "reversing orientation": $-\overline{24} = \overline{42}$



Similarly define the vertex space $\mathbb{Q}V = \mathbb{Q}$-vs with basis V. Typical elts look like $-2\cdot\bar{1} + 3\cdot\bar{2} + \frac{5}{3}\bar{3} - \bar{4}$.

We now define a boundary operator ∂ by

$$\begin{aligned}
\partial : \mathbb{Q}E &\longrightarrow \mathbb{Q}V \\
\overline{12} &\longmapsto \bar{2} - \bar{1} \\
\overline{13} &\longmapsto \bar{3} - \bar{1} \\
\overline{23} &\longmapsto \bar{3} - \bar{2} \\
\overline{24} &\longmapsto \bar{4} - \bar{2} \\
\overline{34} &\longmapsto \bar{4} - \bar{3}
\end{aligned}$$

and extending linearly.

The matrix $A = f_\partial$ is

$$\begin{array}{c} \\ \bar{1} \\ \bar{2} \\ \bar{3} \\ \bar{4} \end{array}
\begin{array}{c} \begin{array}{ccccc} \overline{12} & \overline{13} & \overline{23} & \overline{24} & \overline{34} \end{array} \\
\begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

wrt the indicated ordered bases

The kernel of ∂ is called the cycle space of G.

Compute this by G-J red'n of $(A \mid 0)$:

$$\left(\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

pivots

$x_{12} = x_{23} - x_{34}$

$x_{13} = -x_{23} + x_{34}$

x_{23} free

$x_{24} = -x_{34}$

x_{34} free

indices correspond to edge labels.

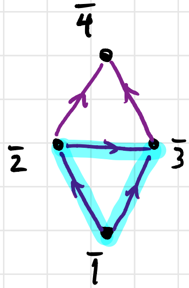
To get a basis, set $(x_{23}, x_{34}) = (1,0)$ and $(0,1)$

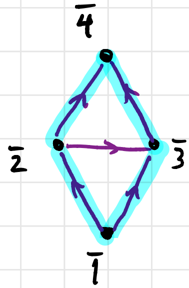
to get

$(1,-1,1,0,0)$ and $(-1,1,0,-1,1)$

respectively, i.e.

$\overline{12} - \overline{13} + \overline{23}$ and $-\overline{12} + \overline{13} - \overline{24} + \overline{34}$





(Note how negative signs correspond to "going backwards.")

What about $\overline{23} - \overline{24} + \overline{34}$? It's the sum of the two ~~basis~~ vectors!

The dimension of the cycle space is called the **cyclomatic number** or **first Betti number** of G. It measures the number of "holes" in G.

(Graphs = 1-dim'l "simplicial complexes." This is a first step towards defining **homology** of such objects.)