Day 17

Learning Goals

- $A, B \in \mathrm{Mat}_{n\times n}(F)$, $AB = I_n \Rightarrow BA = I_n$.
- Matrices $\longleftrightarrow$ linear transformations

Recall last time: Solving

$$\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

was equivalent to row reducing $\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

and getting I_3. To solve

$$\begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix}\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

amounts to row reducing

$$\left(\underbrace{\begin{matrix} 0 & 1 & 1 \\ 3 & 0 & -1 \\ -1 & 1 & 0 \end{matrix}}_{A^T}\ \middle|\ \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right).$$

(check this yourself!)

A^T — transpose of A

Since row rank A = col rank of A, we know $REF(A) = I_n \iff REF(A^T) = I_n$. Thus A has a right inverse iff it has a left inverse.

$$AB = I_n,\quad CA = I_n \implies CAB = I_n B$$
$$CI_n = B$$
$$C = B$$

So left and right inverses are equal.

Linear transformations from matrices:

Given $A \in Mat_{m \times n}(F)$, get a function

$$f_A : F^n \longrightarrow F^m$$
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

which is linear since

$$f_A(u+\lambda v) = A(u+\lambda v) = (Au) + \lambda(Av)$$
$$= f_A(u) + \lambda f_A(v).$$

E.g. Suppose $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. Then

$$f_A : F^3 \longrightarrow F^2$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+3z \\ 4x+5y+6z \end{pmatrix}.$$

i.e. $f_A(x,y,z) = (x+2y+3z, 4x+5y+6z)$.

<u>Note</u>

- Coefficients of components correspond to rows of matrix
- $f_A(e_1) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

  $f_A(e_2) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

  $f_A(e_3) = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

  Columns of A correspond to image of standard basis.

Matrices from linear transformations:

Suppose $f : V \longrightarrow W$ a linear trans between finite-dim'l vector spaces with ordered bases $\alpha = (v_1, \ldots, v_n)$, $\beta = (w_1, \ldots, w_m)$ resp.

We get

$$\begin{tikzcd} F^n \arrow[r, dashed] & F^m \\ V \arrow[u, "\mathrm{Rep}_\alpha", "\cong"'] \arrow[r, "f"] & W \arrow[u, "\mathrm{Rep}_\beta"', "\cong"] \end{tikzcd}$$

$\mathrm{Rep}_\beta \circ f \circ \mathrm{Rep}_\alpha^{-1}$ is the unique linear trans making this diagram commute

What does this do to $e_j \in F^n$?

$$\begin{tikzcd} e_j \arrow[r, mapsto, dashed] \arrow[d, mapsto] & \mathrm{Rep}_\beta f(v_j) \\ v_j \arrow[r, mapsto] & f(v_j) \arrow[u, mapsto] \end{tikzcd}$$

This is the linear trans represented by the matrix A_α^β with j-th column $\mathrm{Rep}_\beta f(v_j)$.

In other words, if

$$f(v_j) = a_{1j} w_1 + a_{2j} w_2 + \cdots + a_{mj} w_m$$

then $A_\alpha^\beta = (a_{ij})$.

E.g. Take $f : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}[x]_{\leq 3}$

$$p \longmapsto xp + p',$$

$\alpha = \langle 1, x, x^2 \rangle$, $\beta = \langle 1, x, x^2, x^3 \rangle$.

We have

$$f(1) = x$$

$$f(x) = x^2 + 1$$

$$f(x^2) = x^3 + 2x^2 .$$

Thus

$$A_\alpha^\beta = \begin{array}{c|ccc} & 1 & x & x^2 \\ \hline 1 & 0 & 1 & 0 \\ x & 1 & 0 & 0 \\ x^2 & 0 & 1 & 2 \\ x^3 & 0 & 0 & 1 \end{array} .$$

TPS Let $V = \operatorname{span}\{\cos, \sin\} \leq \mathbb{R}^{\mathbb{R}}$.

Check that $\frac{d}{dx}$ is a linear trans $V \to V$ and determine its matrix wrt $\langle \cos, \sin \rangle =: \alpha$

Answer $\frac{d}{dx}$ is linear on diff'l functions and

$$\frac{d}{dx}\cos(x) = -\sin(x), \quad \frac{d}{dx}\sin(x) = \cos(x).$$

Thus $A_\alpha^\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.