Day 16

Learning Goals

- Algebraic properties of $\mathrm{Mat}_{n\times n}(F)$
- Defn of inverse matrices
- $\exists$ inverse $\Leftrightarrow$ full rank $\Leftrightarrow$ RREF $= I_n$
- Computation of inverses

Recall the field axioms:

$(R, +, \cdot)$ a field means

$+$ is associative, commutative, has an identity 0, and R has additive inverses: $a + (-a) = 0$

$\cdot$ is associative, commutative, has an identity 1, and $R \setminus \{0\}$ has multiplicative inverses:

$$a \cdot a^{-1} = 1 \text{ for } a \in R \setminus \{0\}$$

- distributes (on either side) over $+$

$(\mathrm{Mat}_{n\times n}(F), +, \cdot)$ has all the same properties

<u>except</u>:
- $\cdot$ is not commutative
- many nonzero matrices do not have inverses

Note that $0_{n\times n} = 0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ is the identity for $+$, and $I_n = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ is the identity for $\cdot$.

E.g. $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

so neither of these matrices have inverses.

(If $AB = 0$, we call A, B zero divisors.)

Write $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$ for the $n\times n$ matrix

$$\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}.$$

Then $I_n = \mathrm{diag}(\underbrace{1, 1, \dots, 1}_{n \text{ terms}})$.

For $A \in \mathrm{Mat}_{m\times n}(F)$, $AI_n = A$.

$B \in \mathrm{Mat}_{n\times m}(F)$, $I_n B = B$.

(Slightly more than identity for $\cdot$ on $\mathrm{Mat}_{n\times n}(F)$.)

Defn For $A \in Mat_{n \times m}(F)$, $B \in Mat_{m \times n}(F)$, when $AB = I_n$, call A a left inverse of B and B a right inverse of A.

E.g. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$AB = I_2$, but $BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \neq I_3$, so

A is left inverse to B but not right inverse.

(In fact, B has no right inverse.)

Prop For $A, B \in Mat_{n \times n}(F)$, $AB = I_n$ iff $BA = I_n$.

(We will prove this next lecture!)

$\boxed{\text{Write } B = A^{-1}.}$

Thm For $A \in Mat_{n \times n}(F)$, TFAE:

(1) A is invertible

(2) $\operatorname{rank}(A) = n$

(3) $REF(A) = I_n$

The proof will follow from our algorithm — about to be presented — for inverting matrices.

Note that we already know (2) $\iff$ (3) from our work on row spaces and rank.

Calculating the inverse (extended G-J)

Let $A = \begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$. To produce an inverse, we need to solve

$$\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Equivalently,

$$\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ d \\ g \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

i.e. $0a + 3d - g = 1$
$1a + 0d + 1g = 0$
$1a - d + 0g = 0$

$$\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} b \\ e \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

i.e. $0b + 3e - 1h = 0$
$1b + 0e + 1h = 1$
$1b - 1e + 0h = 0$

$$\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} c \\ f \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

i.e. $0c + 3f - 1i = 0$
$1c + 0f + 1i = 0$
$1c - 1f + 0i = 1$

Thus we need to perform G-J reduction to

$$\left(\begin{array}{ccc|c} 0 & 3 & -2 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right), \left(\begin{array}{ccc|c} 0 & 3 & -2 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{array}\right), \left(\begin{array}{ccc|c} 0 & 3 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right).$$

The same row ops are required to put the non-augmented piece in REF in each case, so we can work with the "super-augmented" matrix

$$\left(\begin{array}{ccc|ccc} 0 & 3 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right).$$

er -augmented matrix calculation:

$$\left(\begin{array}{ccc|ccc} 0 & 3 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_3 \to r_3 - r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 1 \end{array}\right)$$

$$\xrightarrow[r_3 \to -r_3]{r_2 \leftrightarrow r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 3 & -1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{r_3 \to r_3 - 3r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & -4 & 1 & -3 & 3 \end{array}\right)$$

$$\xrightarrow{r_3 \to -r_3/4} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1/4 & 3/4 & -3/4 \end{array}\right)$$

$$\xrightarrow[r_2 \to r_2 - r_3]{r_1 \to r_1 - r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 1/4 & 3/4 \\ 0 & 1 & 0 & 1/4 & 1/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & 3/4 & -3/4 \end{array}\right).$$

Thus $\begin{pmatrix} a \\ d \\ g \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/4 \\ -1/4 \end{pmatrix}$, $\begin{pmatrix} b \\ e \\ h \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/4 \\ 3/4 \end{pmatrix}$, $\begin{pmatrix} c \\ f \\ i \end{pmatrix} = \begin{pmatrix} 3/4 \\ -1/4 \\ -3/4 \end{pmatrix}$

solve the system. I.e.

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -1 \\ -1 & 3 & -3 \end{pmatrix}.$$

This procedure works in general:

- For $A \in \mathrm{Mat}_{n \times n}(F)$, for the superaugmented matrix $(A \mid I_n) \underset{G\text{-}J}{\rightsquigarrow} (\mathrm{REF}(A) \mid B)$.
- If $\mathrm{REF}(A) = I_n$, then $B = A^{-1}$.
- If $\mathrm{REF}(A) \neq I_n$, then $\mathrm{REF}(A)$ has a row of all 0's. B does not (since $B \underset{\text{row ops}}{\longleftrightarrow} I_n$ so $\mathrm{rank}(B) = n$) so the system is inconsistent and A has no inverse.