Day 15

Learning goals

- Review $\mathrm{Mat}_{m\times n}(F)$ as F-vs
- Define matrix product
$$\mathrm{Mat}_{m\times p}(F) \times \mathrm{Mat}_{p\times n}(F) \longrightarrow \mathrm{Mat}_{m\times n}(F)$$
- Properties of matrix product

Recall $\mathrm{Mat}_{m\times n}(F) = \{m\times n \text{ matrices with entries in } F\}$.

For $A \in \mathrm{Mat}_{m\times n}(F)$, write A_{ij} or $A_{i,j}$ for the entry of A in row i, column j:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1j} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2j} & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3j} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{i1} & A_{i2} & A_{i3} & \cdots & A_{ij} & \cdots & A_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mj} & \cdots & A_{mn} \end{pmatrix}.$$

$\mathrm{Mat}_{m\times n}(F)$ carries a linear structure with "coordinatewise" add'n and scalar mult'n:

$$(A+B)_{ij} := A_{ij} + B_{ij}$$

$$(\lambda A)_{ij} := \lambda A_{ij}.$$

Defn For $A \in \mathrm{Mat}_{m\times p}(F)$, $B \in \mathrm{Mat}_{p\times n}(F)$, the *matrix product* AB is the $m\times n$ matrix with entries

$$(AB)_{ij} = \sum_{k=1}^{p} A_{ik} B_{kj}.$$

"dot product" of i-th row of A, j-th column of B

E.g.

$$\underset{2\times 3}{\begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \end{pmatrix}} \cdot \underset{3\times 2}{\begin{pmatrix} 1 & 0 \\ 3 & -1 \\ 5 & 3 \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} 11 & 6 \\ 26 & 11 \end{pmatrix}}$$

Here 6 = "dot product" of $(1,0,2)$, $(0,-1,3)$

$$= 1\cdot 0 + 0\cdot(-1) + 2\cdot 3.$$

Prop For $A \in \mathrm{Mat}_{m\times n}(F)$, $B \in \mathrm{Mat}_{n\times r}(F)$, $\lambda \in F$

(1) $\lambda(AB) = (\lambda A)B = A(\lambda B)$

(2) $(AB)C = A(BC)$ $\forall C \in \mathrm{Mat}_{r\times s}(F)$

(3) $A(B+C) = AB + AC \quad \forall\, C \in Mat_{n\times r}(F)$

(4) $(C+D)A = CA + DA \quad \forall\, C, D \in Mat_{r\times m}(F)$

Pf of (2) Fix $C \in Mat_{r\times s}(F)$. Then

$$
\begin{aligned}
(A(BC))_{ij} &= \sum_{k=1}^{n} A_{ik} (BC)_{kj} \\
&= \sum_{k=1}^{n} A_{ik} \left(\sum_{l=1}^{r} B_{kl} C_{lj} \right) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{r} A_{ik} (B_{kl} C_{lj}) \\
&= \sum_{l=1}^{r} \sum_{k=1}^{n} A_{ik} (B_{kl} C_{lj}) \\
&= \sum_{l=1}^{r} \sum_{k=1}^{n} (A_{ik} B_{kl}) C_{lj} \\
&= \sum_{l=1}^{r} \left(\sum_{k=1}^{n} A_{ik} B_{kl} \right) C_{lj} \\
&= \sum_{l=1}^{r} (AB)_{il} C_{lj} \\
&= ((AB)C)_{ij}. \quad \square
\end{aligned}
$$

2 Matrix mult'n is <u>not</u> commutative in gen'l.

(a) Dimensions don't match for non-square matrices, in which case product is not even defined!

(b) Even in square case,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

<u>TPS</u> Find all $A \in Mat_{2\times 2}(F)$ such that $\forall B \in Mat_{2\times 2}(F)$, $AB = BA$.

<u>Sol'n</u>

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$$

so need a,b,c,d such that

$$ae+bg = ae+cf, \quad af+bh = be+df$$

$$ce+dg = ag+ch, \quad cf+dh = bg+dh$$

for all $e,f,g,h \in F$. Thus $bg = cf$ and taking $g = f = 1$ implies $b = c$. If $f = 0$, get $bh = be$ $\forall h, e \in F$, so we must have $\boxed{b = 0 = c}$.

But now $af = df$ $\forall f \in F$, so $\boxed{a = d}$.

Check at $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ does indeed commute $\forall a \in F$.

so these "constant diagonal" matrices constitute the "center" of $Mat_{2\times 2}(F)$.

fancy name for matrices commuting with all others

Preview A linear trans $f : F^n \to F^m$ is determined by $f(e_1) = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}$, $f(e_2) = \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix}$, ..., $f(e_n) = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$.

The matrix $\begin{pmatrix} | & & | \\ f(e_1) & \cdots & f(e_n) \\ | & & | \end{pmatrix}$ is a "code" for f.

Given $g : F^m \to F^r$, the matrix product

$$\begin{pmatrix} g(e_1) & \cdots & g(e_m) \end{pmatrix} \begin{pmatrix} f(e_1) & \cdots & f(e_n) \end{pmatrix}$$

is equal to the "code" for $g \circ f : F^n \longrightarrow F^r$.

(We will prove this later, but you can try to do it yourself now!)

No wonder matrix mult'n is associative: so is composition of linear transformations!