Day 14

Learning goals

- Define affine subspaces
- Specify affine subspaces with equations and parametrizations
- Lines, planes, and hyperplanes as special cases.

Dfn Let V, W be $\mathbb{F}$-vector spaces. An affine subspace of V is a set of the form

$$A = p + U = \{p + u \mid u \in U\}$$

where p is some fixed point of V and $U \leq V$.
The dimension of A is the dimension of its linear part, $\dim U$. If $\dim A = k$, we call A a k-plane in V. If $k=1$, call A a line; if $k=2$, a plane, if $k = \dim(V) - 1$, a hyperplane.
A function $\ell : V \longrightarrow W$ is an affine function if $\exists$ lin trans $f : V \longrightarrow W$ s.t. $\ell(v) = p + f(v)$ for some fixed $p \in W$.

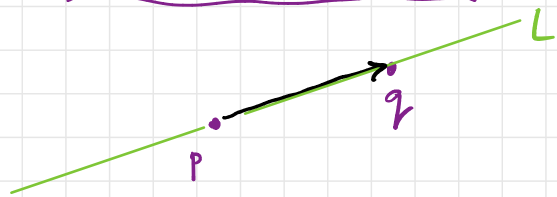
Slogan Affine = linear + constant

We will specify affine subspaces A of dim'n k via

- equations: a system of $n-k$ linear equations with sol'n set A
- parametrizations: an affine function $\ell : F^r \to V$ with image A (often with ℓ injective and $r=k$).

Lines Given distinct points $p, q \in F^n$, the line through p and q is given by

$$\boxed{L = p + \operatorname{span}\{q-p\}}$$

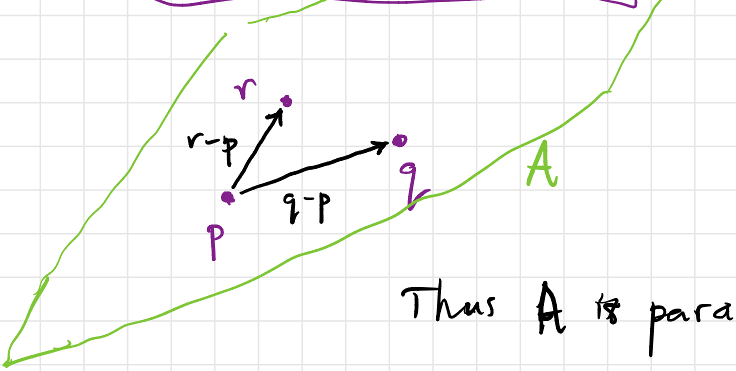


Thus L is parametrized by

$$\boxed{\begin{aligned} \ell : F &\longrightarrow F^n \\ t &\longmapsto p + t(q-p) = (1-t)p + tq \end{aligned}}.$$

Planes Given points $p, q, r \in F^n$ not on a line, the plane containing p, q, r is

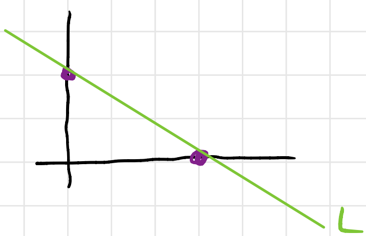
$$\boxed{A = p + \operatorname{span}\{q-p, r-p\}}.$$



Thus A is parametrized by

$$\boxed{\begin{aligned} \ell : F^2 &\longrightarrow F^n \\ (s,t) &\longmapsto p + s(q-p) + t(r-p) \\ &\qquad = (1-s-t)p + sq + tr. \end{aligned}}$$

E.g. The line L in $\mathbb{R}^2$ through $(0,2)$ and $(3,0)$:



Have

$$L = (0,2) + \operatorname{span}\{(3,-2)\}$$

L is specified by an equation of the form

$$ax + by = c.$$

Plug in $(0,2), (3,0)$ to get the system

$$2b = c$$
$$3a = c$$

To get a particular sol'n, plug in $c=6$ to get the particular solution $b=3$, $a=2$, whence L is specified by $2x+3y=6$.

TPS What other equations specify L?

Since $L = (0,2) + \operatorname{span}\{(3,-2)\}$, have the parametrization

$$\begin{aligned} \ell : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (0,2) + t(3,-2) = (3t, 2-2t). \end{aligned}$$

E.g. Consider the line $L \subseteq \mathbb{R}^3$ passing through $(1,2,3)$ and $(-2,4,0)$. Then

$$L = (1,2,3) + \operatorname{span}\{(-3,2,-3)\}$$

and is parametrized by

$$\begin{aligned} \ell : \mathbb{R} &\longrightarrow \mathbb{R}^3 \\ t &\longmapsto (1,2,3) + t(-3,2,-3) \\ &= (1-3t, 2+2t, 3-3t). \end{aligned}$$

L is specified by two linear equations of the form $ax + by + cz = d$. Plugging in $(1,2,3)$ and $(-2,4,0)$, get the system

$$a + 2b + 3c = d$$
$$-2a + 4b \qquad = d$$

i.e.

$$a + 2b + 3c - d = 0$$
$$-2a + 4b \qquad - d = 0.$$

Apply G-J red'n:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 0 \\ -2 & 4 & 0 & -1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 3/2 & -1/4 & 0 \\ 0 & 1 & 3/4 & -3/8 & 0 \end{array}\right).$$

Thus

$$a = -\frac{3}{2}c + \frac{1}{4}d$$
$$b = -\frac{3}{4}c + \frac{3}{8}d.$$

Free variables c, d.

Set $(c,d) = (1,0)$ and $(0,1)$ to get basis

$$\left(-\frac{3}{2}, -\frac{3}{4}, 1, 0\right), \left(\frac{1}{4}, \frac{3}{8}, 0, 1\right)$$

of sol'n space. Scale to get basis

$$(-6, -3, 4, 0), (2, 3, 0, 8)$$

which is prettier.

We get eq'ns

$$-6x - 3y + 4z = 0$$
$$2x + 3y \qquad = 8$$

specifying L. Each corresponds to a plane and their intersection is L.

E.g. Let P be the plane in $\mathbb{R}^3$ passing through $(0,2,-1)$, $(4,2,1)$, and $(1,0,1)$. Then

$$P = (0,2,-1) + \operatorname{span}\{(4,0,2),(1,-2,2)\}$$ and is

parametrized by

$$\ell : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(s,t) \longmapsto (0,2,-1) + s(4,0,2) + t(1,-2,2)$$
$$= (4s+t,\ 2-2t,\ -1+2s+2t).$$

P is specified by the linear eq'n $ax+by+cz=d$ which is satisfied by $(0,2,-1)$, $(4,2,1)$, and $(1,0,1)$. This gives the system

$$\begin{aligned} 2b - c - d &= 0 \\ 4a + 2b \qquad - d &= 0 \\ a \qquad + c - d &= 0 \end{aligned}$$

Apply G-J red'n:

$$\begin{pmatrix} 0 & 2 & -1 & -1 \\ 4 & 2 & 0 & -1 \\ 1 & 0 & 1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

Thus $a = -d$, $b = 3/2\, d$, $c = 2d$.

Setting $d = 2$ gives

$$-2x + 3y + 4z = 2$$

as the eq'n for P.