Day 13

Learning Goals

- $\ker(f) = 0$ detects injectivity
- Define isomorphism
- Dimension detects isomorphism

Throughout, $f: V \to W$ a linear transformation.

Thm f injective $\Leftrightarrow$ $\ker(f) = \{0\}$.

Pf ($\Rightarrow$) Since f is linear, $f(0) = 0$, so $0 \in \ker(f)$.

As f is injective, no other vectors map to 0 under f so $\ker(f) = \{0\}$.

($\Leftarrow$) Suppose $f(v) = f(w)$. Then

$$0 = f(v) - f(w) = f(v-w).$$

Thus $v - w \in \ker(f) = \{0\}$ so $v - w = 0$, i.e. $v = w$.

This means f is injective. $\square$

Prop For $S \subseteq V$,

(1) S lin dependent $\Rightarrow$ $fS = \{f(s) \mid s \in S\}$ lin dep

(2) S lin ind $+$ f injective $\Rightarrow$ fS lin ind.

TPS Find f, S lin ind s.t. fS is not lin ind.

Pf of Prop (1) Suppose $\sum \lambda_i s_i = 0$ for $s_i \in S$, $\lambda_i \in F$. Applying f, $\sum \lambda_i f(s_i) = 0$ so f preserves linear dependencies, and fS is lin dep.

(2) Fix $f(s_1), \dots, f(s_n) \in fS$ and suppose $\sum \lambda_i f(s_i) = 0$. By linearity of f, $f(\sum \lambda_i s_i) = 0$. Since f is injective, the theorem implies $\sum \lambda_i s_i = 0$. But S is lin ind, so $\lambda_i = 0$ $\forall i$. Hence fS is lin ind. $\square$

Defn A linear transformation $f: V \to W$ is an *isomorphism* when $\exists$ lin trans $g: W \to V$ s.t. $g \circ f = id_V$, $f \circ g = id_W$. Call V, W *isomorphic* and write $V \cong W$.

Rmk It follows that f is a bijection. You should check that all linear bij'ns are isomorphisms. (They automatically have an inverse function, so you need to check that the inverse is linear.)

E.g. • $\mathrm{Mat}_{2\times 2}(F) \cong F^4$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d)$

• $F[x] \cong \mathbb{F}^{\mathbb{N}}$ via $\lambda_0 + \lambda_1 x + \lambda_2 x^2 + \dots \mapsto (i \mapsto \lambda_i)$

TPS Check that $\cong$ is an equivalence relation.

Prop $f: V \to W$ is an isomorphism iff $\ker(f) = \{0\}$ and $\mathrm{im}(f) = W$.

Pf The second condition is the defn of surjectivity, and the first is equiv to injectivity by the theorem.

Thm If V, W are finite dimensional, then

$$V \cong W \quad \text{iff} \quad \dim V = \dim W.$$

Pf We first prove that $\dim V = n < \infty \Rightarrow V \cong F^n$:

Choose a basis $\{b_1, \ldots, b_n\}$ of V and define a lin trans $f: V \to F^n$ via $f(b_i) = e_i$, i.e.

$f = \mathrm{Rep}_B : \sum \lambda_i b_i \mapsto (\lambda_1, \ldots, \lambda_n)$. Since $\{e_1, \ldots, e_n\}$ is a basis of F^n, may define $g: F^n \to V$ by $g(e_i) = b_i$. Check: $g \circ f = id_V$, $f \circ g = id_V$. (Suffices to check on bases!) Thus $V \cong F^n$.

Now suppose $\dim V = \dim W = n < \infty$. Then $V \cong F^n$, $W \cong F^n$, and $\cong$ is an equiv rel'n, so $V \cong W$.

Finally, suppose $V \cong W$, both finite dimensional.

Check: iso sends basis to basis, so $\dim V = \dim W$.

$\square$

Rmk Specifying an iso $V \to F^n$ is equivalent to choosing a basis of V b/c the inverse iso $g: F^n \to V$ gives $g(e_1), \ldots, g(e_n)$ as a basis.