Day 12

Learning Goals

- Define image and kernel of linear trans'ns
- Rank and nullity as dimns of these subspaces
- Rank-nullity theorem

Defn Given a linear trans $f: V \to W$, the *image* of f is $\operatorname{im}(f) = fV = \{f(v) \mid v \in V\}$.

Prop $\operatorname{im}(f) \leq W$.

Pf (1) $0 = f(0) \in \operatorname{im}(f)$

(2) if $f(u), f(v) \in \operatorname{im}(f)$, then $f(u)+f(v) = f(u+v) \in \operatorname{im}(f)$.

(3) if $f(u) \in \operatorname{im}(f)$, then $\lambda f(u) = f(\lambda u) \in \operatorname{im}(f)$. $\square$

Defn The *rank* of a linear tran'n $f: V \to W$ is

$$\operatorname{rank}(f) := \dim \operatorname{im}(f).$$

E.g. Define a linear trans $f: \mathbb{R}^2 \to \mathbb{R}^3$ by $f(1,0) = (2,1,0)$, $f(0,1) = (0,-1,1)$ and extending linearly. (This means

$$f(x,y) = x(2,1,0) + y(0,-1,1)$$
$$= (2x, x-y, y).$$

Then $\mathrm{im}(f) = \mathrm{span}\{(2,1,0), (0,-1,1)\}$ and $(2,1,0)$, $(0,-1,1)$ are lin ind (why?) so $\mathrm{rank}(f) = 2$.

Rmk If $f: V \to W$ linear and $\{b_1, \ldots, b_n\}$ is a basis of V, then $\mathrm{im}(f) = \mathrm{span}\{f(b_1), \ldots, f(b_n)\}$.
If $\{f(b_1), \ldots, f(b_n)\}$ not lin ind, then have $\mathrm{rank}(f) < n$.

Defn If $f: V \to W$ linear and $U \leq W$, then the **preimage** of U under f is
$$f^{-1}U := \{v \in V \mid f(v) \in U\}.$$

Prop For $f: V \to W$ linear and $U \leq W$, $f^{-1}U \leq V$.

Pf (1) $0 \in U$ and $f(0) = 0$ so $0 \in f^{-1}U$.

(2) if $u, v \in f^{-1}U$ then $f(u+v) = f(u) + f(v) \in U$ since $f(u), f(v) \in U$ which is closed under add'n. Thus $u+v \in f^{-1}U$.

(3) if $u \in f^{-1}U$ then $f(\lambda u) = \lambda f(u) \in U$ so $\lambda u \in f^{-1}U$. $\square$

Defn For $f: V \to W$ linear, the kernel (or nullspace) of f is $\ker(f) = f^{-1}\{0\} = \{v \in V \mid f(v) = 0\}$.

Note $\ker(f) \leq V$.

The nullity of f is $\dim \ker(f)$.

TPS What is $\ker\left(f: \mathbb{R}^2 \to \mathbb{R}^3\right)$?

$$(x,y) \mapsto (2x, x-y, y)$$

E.g. Consider the linear trans

$$f: \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}^2$$

$$a+bx+cx^2 \longmapsto (a+b, a+c).$$

To find $\ker(f)$ need to solve $f(a+bx+cx^2) = (0,0)$.

This is equiv to
$$a+b=0$$
$$a+c=0.$$

Apply G-J red'n:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

so $a = -c$, $b = c$ and

$$\ker(f) = \{-c + cx + cx^2 \mid c \in \mathbb{R}\}$$
$$= \operatorname{span}\{-1 + x + x^2\}.$$

Hence nullity of f is $\dim \ker(f) = 1$.

What about rank? $\mathbb{R}[x]_{\leq 2}$ has basis $\{1, x, x^2\}$ and

$f(1) = (1,1)$, $f(x) = (1,0)$, $f(x^2) = (0,1)$ so

$$\operatorname{im}(f) = \operatorname{span}\{(1,1), (1,0), (0,1)\} = \mathbb{R}^2$$

so $\operatorname{rank}(f) = 2$.

Note that $\dim \mathbb{R}[x]_{\leq 2} = 3 = 2+1 = \operatorname{rank}(f) + \operatorname{nullity}(f)$.

This is generic:

THM (rank-nullity) Suppose $f: V \to W$ linear and V is finite dim'l. Then

$$\dim V = \operatorname{rank}(f) + \operatorname{nullity}(f).$$

V "splits" into the piece f kills ($\ker(f)$) and the piece f preserves

Pf Suppose $\operatorname{nullity}(f) = k$ and $\ker(f)$ has basis $K = \{v_1, \ldots, v_k\}$. Complete K to a basis for V,

$$B = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}.$$

To prove the theorem, it suffices to show that

$\{f(v_{k+1}), \dots, f(v_n)\}$ is a basis of $\mathrm{im}(f)$.

Know that

$$\mathrm{im}(f) = \mathrm{span}\, fB = \mathrm{span}\{0, \dots, 0, f(v_{k+1}), \dots, f(v_n)\}$$
$$= \mathrm{span}\{f(v_{k+1}), \dots, f(v_n)\}$$

so this set generates $\mathrm{im}(f)$.

For linear ind, suppose

$$\lambda_{k+1} f(v_{k+1}) + \cdots + \lambda_n f(v_n) = 0.$$

Then $f(\lambda_{k+1} v_{k+1} + \cdots + \lambda_n v_n) = 0$

so $\lambda_{k+1} v_{k+1} + \cdots + \lambda_n v_n \in \ker(f)$. Since K is a basis of $\ker(f)$, $\exists \lambda_1, \dots, \lambda_k \in F$ s.t.

$$\lambda_1 v_1 + \cdots + \lambda_k v_k = \lambda_{k+1} v_{k+1} + \cdots + \lambda_n v_n.$$

But then

$$\lambda_1 v_1 + \cdots + \lambda_k v_k - \lambda_{k+1} v_{k+1} - \cdots - \lambda_n v_n = 0.$$

By lin ind of B, all $\lambda_i = 0$. In particular, $\{f(v_{k+1}), \dots, f(v_n)\}$ is lin ind. $\square$