Day 11

Learning goals

- define linear transformations
- how to prove that a function is linear
- understand basic properties
- the vector space $\mathrm{Hom}(V,W)$ of lin trans $V \to W$.

Objects	Relations b/w objects
Sets	Functions
Open subsets of $\mathbb{R}$	Continuous/diff'l func'ns
Vector spaces	Linear transformations

Aside Contemporary mathematics (esp. "category theory") emphasizes relations over objects.

Defn Fix vector spaces V, W over a field F. A linear transformation $f: V \longrightarrow W$ is a function s.t. $\forall u, v \in V$, $\lambda \in F$,

$$f(u+v) = f(u) + f(v) \quad \text{and} \quad f(\lambda u) = \lambda f(u).$$

f "preserves addition" $\qquad$ f "preserves scalar mult'n"

Diagramatically,

$$\begin{tikzcd} V \times V \arrow[r, "+"] \arrow[d, "f \times f"'] & V \arrow[d, "f"] \\ W \times W \arrow[r, "+"'] & W \end{tikzcd} \qquad \begin{tikzcd} F \times V \arrow[r, "\cdot"] \arrow[d, "id \times f"'] & V \arrow[d, "f"] \\ F \times W \arrow[r, "\cdot"'] & W \end{tikzcd}$$

commute (compositions w/ same source and target are equal).

TPS Show that $f : V \to W$ is linear iff

$$f(u + \lambda v) = f(u) + \lambda f(v) \quad \forall u, v \in V, \lambda \in F.$$

! Hefferon limits "linear transformations" to those w/ equal domain and codomain, $f : V \to V$. This is not standard. Such linear transf'ns are typically called (linear) endomorphisms.

E.g. The function $f : \mathbb{R}^3 \to \mathbb{R}^2$

$$(x, y, z) \mapsto (2x + 3y, x + y - 3z)$$

is linear.

What follows may be treated as a template for proving that a fn is linear.

Pf Let $(x,y,z), (x',y',z') \in \mathbb{R}^3$. Then

$$\begin{aligned} f((x,y,z)+(x',y',z')) &= f(x+x', y+y', z+z') \\ &= (2(x+x')+3(y+y'), (x+x')+(y+y')-3(z+z')) \\ &= ((2x+3y)+(2x'+3y'), (x+y-3z)+(x'+y'-3z')) \\ &= (2x+3y, x+y-3z)+(2x'+3y', x'+y'-3z') \\ &= f(x,y,z)+f(x',y',z'). \end{aligned}$$

Thus f preserves addition. Next, for $\lambda \in \mathbb{R}$,

$$\begin{aligned} f(\lambda(x,y,z)) &= f(\lambda x, \lambda y, \lambda z) \\ &= (2(\lambda x)+3(\lambda y), (\lambda x)+(\lambda y)-3(\lambda z)) \\ &= (\lambda(2x+3y), \lambda(x+y-3z)) \\ &= \lambda(2x+3y, x+y-3z) \\ &= \lambda f(x,y,z). \end{aligned}$$

Thus f also preserves scalar mult'n, so f is a linear trans'n. $\square$

Don't confuse "closed under" (as in subspace verification) with "preserves".

TPS Prove or disprove the following assertion:

$$\begin{aligned} f: \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned} \quad \text{is linear.}$$

(It's not: $f(1+1) = 2^2 = 4$ while $f(1)+f(1) = 1^2+1^2 = 2$.)

Prop 1 If $f: V \to W$ is linear, then $f(0) = 0$.

Pf By linearity,

$$f(0_V) = f(0_F \cdot 0_V) = 0_F \cdot f(0_V) = 0_W. \quad \square$$

THM (A linear trans'n is determined by its action on a basis)

Let V, W be F-vs's and suppose B is a basis of V. For each $b \in B$, suppose $w_b \in W$. Then $\exists!$ lin trans'n $f: V \to W$ such that $f(b) = w_b$ $\forall b \in B$.

Pf Define f as follows. For $v \in V$ write

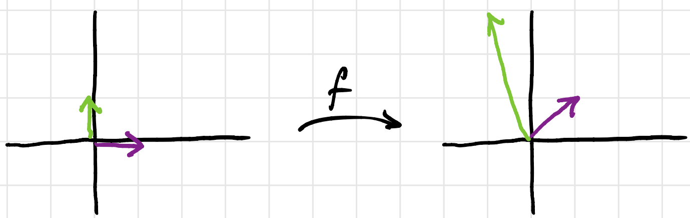
$v = \sum_{i=1}^{n} \lambda_i b_i$ for some $b_i \in B$. (Can do so uniquely since B is a basis.) Define

$$f(v) = \sum_{i=1}^{n} \lambda_i w_{b_i}.$$

This is well-defined by and the expression is forced by linearity + $f(b)=w_b$ for $b\in B$. $\square$

<u>Terminology</u> We say that such f is "defined on B and then extended linearly."

<u>E.g.</u> There is a unique linear trans'n $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$ such that $(1,0)\mapsto(1,1)$, $(0,1)\mapsto(-1,3)$.



We have
$$\begin{aligned} f(x,y) &= f(x(1,0)+y(0,1)) \\ &= x f(1,0) + y f(0,1) \\ &= x(1,1) + y(-1,3) \\ &= (x-y,\ x+3y). \end{aligned}$$

<u>Defn</u> For V, W both F-vs's, let
$$\mathrm{Hom}(V,W) := \{ f: V\to W \mid f \text{ linear} \}.$$
This is also denoted $\mathrm{Hom}_F(V,W)$ or $\mathcal{L}(V,W)$.

The set $Hom(V,W)$ has a linear structure via

$$\begin{aligned} f+\lambda g : V &\longrightarrow W \\ v &\longmapsto f(v)+\lambda f(g) \end{aligned}$$

for $f,g \in Hom(V,W)$, $\lambda \in F$.