Day 10

Learning Goals

- define row and column spaces and rank of a matrix
- Use G-J red'n to compute both
- Prove row and col rank equal $\rightsquigarrow$ <u>the</u> rank of a matrix

Defn $A \in Mat_{m\times n}(F)$. The row space of A is the subspace of F^n generated by the rows of A, and the column space of A is the subspace of F^m generated by its columns. The row rank of A is the dimension of its row space, the column rank of A is the dimension of its column space.

Recall The elementary row operations:

- are linear combos / reorderings of rows
- are reversible.

Thus the row space of A = the row space of its reduced echelon form.

Given the shape of REF, we in fact have that

the nonzero rows of REF(A) form a basis of the row space of A.

E.g. Let $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 1 & 0 \\ 7 & 8 & 2 & 4 \end{pmatrix} \rightsquigarrow \text{REF}(A) = \begin{pmatrix} 1 & 0 & 2/3 & -4 \\ 0 & 1 & -1/3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Thus a basis for the row space of A is

$\{(1,0,2/3,-4), (0,1,-1/3,4)\}$; the row rank of A is 2.

Prop Let $A = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix} \in \text{Mat}_{m\times n}(F)$ with

c_i = column vectors in F^m. Let $\tilde{A}$ be any matrix formed by doing row ops to A, and let $\tilde{c}_1, \ldots, \tilde{c}_n$ be its columns. Then for $\lambda_1, \ldots, \lambda_n \in F$,

$$\sum_{i=1}^n \lambda_i c_i = 0 \iff \sum_{i=1}^n \lambda_i \tilde{c}_i = 0.$$

Pf Write out the first eq'n:

$$\lambda_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + \lambda_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = 0.$$

Its solutions are those of the system of linear eq'ns

$$a_{11}\lambda_1 + a_{12}\lambda_2 + \cdots + a_{1n}\lambda_n = 0$$
$$\vdots$$
$$a_{m1}\lambda_1 + a_{m2}\lambda_2 + \cdots + a_{mn}\lambda_n = 0.$$

Row ops don't change sol'n sets, so the result follows. $\square$

Cor Let $E = REF(A)$ and suppose the pivot columns have indices $j_1, \ldots, j_r$. Then the columns of A indexed by $j_1, \ldots, j_r$ form a basis of the column space of A.

Pf For simplicity, assume $j_1 = 1, \ldots, j_r = r$. Then E looks like

$$\begin{pmatrix} 1 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

in the $m=5, n=6, r=3$ case.

Let $E_1, \ldots, E_n$ denote the col's of E, $A_1, \ldots, A_n$ the col's of A. Need to show $A_1, \ldots, A_r$ are lin ind and generate the column space.

For linear ind, suppose $\lambda_1 A_1 + \cdots + \lambda_r A_r = 0$.

By the prop, $\lambda_1 E_1 + \cdots + \lambda_r E_r = 0$. But

$E_1, \ldots, E_r$ are lin ind, $\lambda_1, \ldots, \lambda_r = 0$.

To show that $\text{span}\{A_1, \ldots, A_r\} = $ col space of A,

it suffices to show $A_j \in$ for $j > r$. Since

$E_1, \ldots, E_r$ span the column space of E (check this!)

know $\exists\, \lambda_1, \ldots, \lambda_r \in F$ s.t. $\lambda_1 E_1 + \cdots + \lambda_r E_r = E_j$.

Equivalently, $\lambda_1 E_1 + \cdots + \lambda_r E_r - E_j = 0$

so, by prop, $\lambda_1 A_1 + \cdots + \lambda_r A_r - E_j = 0$

$\iff \lambda_1 A_1 + \cdots + \lambda_r A_r = A_j$. $\square$

E.g. $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 1 & 0 \\ 7 & 8 & 2 & 4 \end{pmatrix} \rightsquigarrow \text{REF}(A) = \begin{pmatrix} 1 & 0 & 2/3 & -4 \\ 0 & 1 & -1/3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

so $\left\{ \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} \right\}$ is a basis for the col space of A

and the column rank of A is 2.

Thm Row rank of A = Col rank of A.

Pf Let $E = REF(A)$. The number of nonzero rows of A = # pivot columns. $\square$

Defn The **rank** of A, denoted $\operatorname{rank}(A)$, is its row rank or column rank.

Rank detects unique solutions:

To compute sol'ns of a homogeneous system

$$a_{11}x_1 + \cdots + a_{1n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = 0$$

we compute $REF(A)$ for $A = (a_{ij})$. The number of free variables is the number of non-pivot columns $= n - \operatorname{rank}(A)$. $\boxed{\exists!\text{ sol'n } \vec{0} \text{ iff } \operatorname{rank}(A) = n.}$

For a non-homogeneous system

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

We compute REF($[A|b]$) for $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$.

If the system is consistent, then every sol'n is of the form (particular sol'n) + (sol'n of corresponding homogeneous system).

So if the system is consistent, there is again a unique sol'n $\Leftrightarrow$ rank$(A) = n$.