Day 9

Learning goals

- prove that the dimension of a (finite dimensional) vector space is well-defined
- compute $\dim V$

Defn A vector space is finite dimensional if it has a basis with finitely many elements.

E.g.: • F^n, $\mathrm{Mat}_{m\times n}(F)$ are finite dimensional with bases of cardinality $n, mn < \infty$ resp.

• $F[x]$, $\mathbb{R}^{\mathbb{R}}$ are infinite dimensional (no finite basis).

We will prove today that if V is finite dimensional and B, B' are bases of V, then $|B| = |B'|$. This will allow us to make the following defn:

Defn If V is a finite dimensional F-vs, then the dimension of V, denoted $\dim V$ or $\dim_F V$ is the cardinality of any basis of V.

Exchange Lemma Suppose $B = \{v_1, \dots, v_n\}$ is a basis of V and $w = \sum \lambda_i v_i \in V$ (⊛) with $\lambda_i \in F$ not all 0. If $\lambda_\ell \neq 0$ for some $\ell \in \{1, \dots, n\}$, then

$$B' = (B \setminus \{v_\ell\}) \cup \{w\}$$ ← exchange v_ℓ and w

is also a basis of V.

Pf First show B' lin ind. WLOG, $\ell = 1$.

Suppose $\mu w + \mu_2 v_2 + \dots + \mu_n v_n = 0$. Since $w = \sum \lambda_i v_i$ (⊛), substituting gives

$$\mu \left(\sum_{i=1}^{n} \lambda_i v_i \right) + \mu_2 v_2 + \dots + \mu_n v_n = 0$$

$$\Leftrightarrow \quad \mu \lambda_1 v_1 + (\mu \lambda_2 + \mu_2) v_2 + \dots + (\mu \lambda_n + \mu_n) v_n = 0$$

Since the v_i are lin ind, we have

$$\mu \lambda_1 = \mu \lambda_2 + \mu_2 = \dots = \mu \lambda_n + \mu_n = 0$$

Since $\lambda_1 \neq 0$, know $\mu = 0$, whence

$$\mu_2 = \dots = \mu_n = 0$$

Therefore $B' = \{w, v_2, \dots, v_n\}$ is lin ind.

Now show that $\operatorname{span} B' = V$. Solving for v_1 in (⊛) gives

$$\textcircled{†} \quad v_1 = \frac{1}{\lambda_1} w - \frac{\lambda_2}{\lambda_1} v_2 - \cdots - \frac{\lambda_n}{\lambda_1} v_n .$$

Given $v \in V$, use the fact that B is a basis to write

$$v = \mu_1 v_1 + \cdots + \mu_n v_n \text{ for some } \mu_i \in F.$$

Substing † in gives

$$v = \mu_1 \left(\frac{1}{\lambda_1} w - \frac{\lambda_2}{\lambda_1} v_2 - \cdots - \frac{\lambda_n}{\lambda_1} v_n \right) + \mu_2 v_2 + \cdots + \mu_n v_n$$

$$= \frac{\mu_1}{\lambda_1} w + \left(\mu_2 - \frac{\mu_1 \lambda_2}{\lambda_1}\right) v_2 + \cdots + \left(\mu_n - \frac{\mu_1 \lambda_n}{\lambda_1}\right) v_n$$

$$\in \operatorname{span} B' .$$

Thus B' generates $V \Longrightarrow B'$ is a basis. $\square$

Thm In a finite dim'l vector space V, every basis has the same cardinality.

Pf Among all bases of V let $B = \{v_1, v_2, \ldots, v_n\}$ be one of minimal size. Let $C = \{w_1, w_2, \ldots\}$ be another basis of V. WTS $|C| = |B|$. By the choice of B, know $|C| \geq |B|$.

Idea: use the exchange lemma to swap n elts of C into B, maintaining basis status.

Let $B_0 = B$ and take $w_1 \in C$. By the exchange lemma, get a new basis B_1 by swapping w_1 in for some $v_\ell \in B_0$. WLOG, $\ell = 1$ and

$$B_1 = \{w_1, v_2, \dots, v_n\} \text{ is a basis.}$$

Since B_1 is a basis, have

$$w_2 = \lambda_1 w_1 + \lambda_2 v_2 + \dots + \lambda_n v_n \quad , \quad \lambda_i \in F.$$

Since w_1, w_2 lin ind, some λ_ℓ, $\ell \geq 2$ is nonzero. WLOG, $\ell = 2$ and swapping w_2 for v_2 gives a basis $B_2 = \{w_1, w_2, v_3, \dots, v_n\}$. Continuing in this fashion, eventually get that

$$\begin{array}{l} B_n = \{w_1, \dots, w_n\} \text{ is a basis.} \\ \cap\!| \\ C \end{array}$$

In fact, $B_n = C$ b/c w_{n+1} would be in span $B_n \Rightarrow C$ lin dep ☹. $\square$

Cor If V is a fin dim'l vector space, then our "basis production algorithm" extends any lin ind set S to a basis of V in $\dim V - |S|$ steps. $\square$

Cor If V is a fin dim'l vector space and $T \subseteq V$ generates V, then $\exists S \subseteq T$ which is a basis of V. $\square$

Cor If $S \subseteq V$ and $|S| = n = \dim V < \infty$, then S is lin ind iff $\operatorname{span} S = V$.

Pf ($\Rightarrow$) If $|S| = n$ and S lin ind, then S can be completed to a basis. This basis cannot have more than n elts, so S is already a basis.

($\Leftarrow$) If $|S| = n$ and $\operatorname{span} S = V$, then there is a subset of S which is a basis, but this basis must have n elts, so S is already a basis. $\square$

E.g. (1) F^n has basis $\{e_1, \dots, e_n\}$ so $\dim F^n = n$.

(2) Let $S = \{(1,0,0), (1,2,0), (1,2,3)\} \subseteq \mathbb{R}^3$.

Since
- $(1,2,0) \notin \operatorname{span}\{(1,0,0)\}$
- $(1,2,3) \notin \operatorname{span}\{(1,0,0), (1,2,0)\}$
- $\dim \mathbb{R}^3 = 3 = |S|$

know S is a basis of $\mathbb{R}^3$.

(7) $F[x]_{\leq 2}$ has basis $\{1, x, x^2\}$ so $\dim F[x]_{\leq 2} = 3$

polynomials of degree ≤ 2

TPS Let $F[x,y] = \left\{ \sum \lambda_{ij} x^i y^j \;\middle|\; \begin{array}{l} \text{sum is finite,} \\ \lambda_{ij} \in F, \\ i,j \geq 0 \end{array} \right\}$

be the F-vs of 2-variable polynomials over F.

Define $\deg\left(\sum \lambda_{ij} x^i y^j\right) = \max\{i+j \mid \lambda_{ij} \neq 0\}$ and let $F[x,y]_{\leq n}$ be the subspace of polynomials of degree $\leq n$.

- What is $\dim F[x,y]_{\leq 3}$?
- $\dim F[x,y]_{\leq n}$?