Day 8

Learning goals:

- introduce bases
- preservation of linear structure (preview of isomorphism)

Defn A subset $B \subseteq V$ is a basis of $V$ when $\operatorname{span} B = V$ and $B$ is linearly independent.

Moral exercise $B \subseteq V$ is a basis iff $B$ is a minimal generating set iff $B$ is a maximal linearly independent set. (Minimal / maximal wrt $\subseteq$.)

Defn An ordered basis is a basis with elts listed as a sequence: $\langle b_1, b_2, \dots \rangle$.

pointy brackets just notation.

In Hefferon, only ordered bases are considered and these are called bases.

**Defn** Given an ordered basis $B = \langle b_1, \dots, b_n \rangle$ of $V$ and $v \in V$, $\exists!\ \lambda_1, \dots, \lambda_n \in F$ s.t.

$$v = \lambda_1 b_1 + \cdots + \lambda_n b_n$$

The coordinates of $v$ with respect to $B$ are the vector $(\lambda_1, \dots, \lambda_n) \in F^n$.

$$\underset{\mathrm{Rep}_B(v)}{\overset{\parallel}{}}$$

standard basis

**E.g.** (1) $F^3$ has ordered basis $B = \langle e_1, e_2, e_3 \rangle$ for $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. Since $(x,y,z) = xe_1 + ye_2 + ze_3$, the coordinates of $(x,y,z)$ are $(x,y,z)$. ☺

(2) Take $B' = \langle e_3, e_2, e_1 \rangle$. Then the coords of $(x,y,z)$ are $(z,y,x)$

— order matters!

(3) Check that $B'' = \langle (1,0,0), (1,1,0), (1,1,1) \rangle$ is a basis of $F^3$. Since

$$(x,y,z) = (x-y)(1,0,0) + (y-z)(1,1,0) + z(1,1,1)$$

get coords of $(x,y,z)$ are $(x-y, y-z, z)$.

For instance, $\mathcal{B}'$ coords of $(1,0,3)$ are $(1,-3,3)$.

(4) TPS Find a basis for $\mathrm{Mat}_{2\times 2}(F)$.

Probably get $\left\langle \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix} \right\rangle$

in which case $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ has coords $(a,b,c,d)\in F^4$.

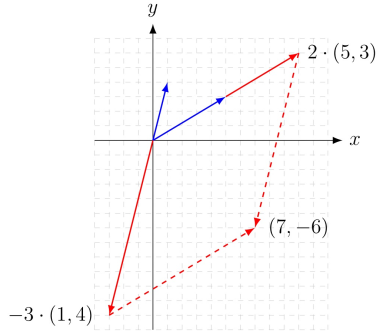
(5) Let's write $(7,-6)\in\mathbb{R}^2$ in coords wrt $\langle (5,3),(1,4)\rangle$ (which is an ordered basis of $\mathbb{R}^2$).

Need $(\lambda,\mu)$ s.t. $\lambda(5,3)+\mu(1,4)=(7,-6)$

$$\iff \begin{aligned} 5\lambda+\mu&=7\\ \lambda+4\mu&=-6\end{aligned}.$$

Aug'd matrix $\left(\begin{array}{cc|c}5&1&7\\1&4&-6\end{array}\right) \overset{GJ}{\rightsquigarrow} \left(\begin{array}{cc|c}1&0&2\\0&1&-3\end{array}\right)$

so coords are $(2,-3)$.



## Towards isomorphism (preservation of linear structure)

Given an ordered basis $B = \langle v_1, \dots, v_n \rangle$ of $V$, get inverse bij'ns

$$V \underset{}{\overset{\mathrm{Rep}_B}{\rightleftarrows}} F^n$$

$$v \longmapsto \mathrm{Rep}_B(v)$$

$$x_1 v_1 + \dots + x_n v_n \longleftarrow\!\!| \ (x_1, \dots, x_n).$$

Even better, these functions preserve linear structure: $f(v+w) = f(v) + f(w)$, $f(\lambda v) = \lambda f(v)$.

Since both elements and operations align, $V$ and $F^n$ are "essentially the same".

After we study linear transformations, we will call $V$ and $F^n$ isomorphic.