Day 7

Learning Goals

- Get super comfy w/ linear independence

F field, V an F-vs.

Recall that $u_1, \dots, u_n \in V$ are distinct **linearly independent** when

$$\sum_{i=1}^{n} \lambda_i u_i = 0 \implies \lambda_i = 0 \text{ for } 1 \le i \le n.$$

Thm Let $S \subseteq V$ be lin ind, $v \in \operatorname{span} S$. Then v has a unique expression as a linear combo of elts of S. To be precise, if

$$v = \sum_{i=1}^{n} \lambda_i u_i = \sum_{j=1}^{m} \mu_j w_j \quad \text{for some}$$

$\lambda_i, \mu_j \in F \setminus \{0\}$, $u_i, w_j \in S$, then $n = m$ and, up to re-indexing, $u_i = w_i$, $\lambda_i = \mu_i \ \forall i$.

Pf WLOG, $v = \sum_{i=1}^{n} \lambda_i u_i = \sum_{i=1}^{n} \mu_i u_i$ by allowing 0 coeffs. Then

$$0 = v - v = \sum_{i=1}^{n} (\lambda_i - \mu_i) u_i \Rightarrow \lambda_i - \mu_i = 0 \ \forall i$$

by lin ind of S. Hence $\lambda_i = \mu_i \ \forall i$. $\square$

NB This is a special property of lin ind sets.

If $S = \{(1,1), (2,2)\} \subseteq \mathbb{R}^2$, then

$$(3,3) = (1,1) + (2,2) = 2(1,1) + \tfrac{1}{2}(2,2)$$

etc.

Fact (moral exc) The converse of the theorem is true.

Idea Create a maximal set of lin ind vectors in V via the following process:

- pick $v_1 \in V \setminus \{0\}$
- if $\operatorname{span}\{v_1\} = V$, stop
- o/w take $v_2 \in V \setminus \operatorname{span}\{v_1\}$
- if $\operatorname{span}\{v_1, v_2\} = V$, stop
- o/w take $v_3 \in V \setminus \operatorname{span}\{v_1, v_2\}$
- ... etc.

Prop If $S \subseteq V$ is lin ind and $v \in V \setminus S$ then $S \cup \{v\}$ is lin dep iff $v \in \operatorname{span} S$.

Pf ($\Rightarrow$) Suppose $S \cup \{v\}$ is lin dep. Then we have a nontrivial lin rel'n

$$\lambda v + \lambda_1 u_1 + \cdots + \lambda_n u_n = 0 \quad (*)$$

for some $\lambda, \lambda_i \in F$ not all 0, distinct $u_i \in S$.
Since S is lin ind, $\lambda \neq 0$. Solve for v:

$$v = \frac{-\lambda_1}{\lambda} u_1 - \cdots - \frac{\lambda_n}{\lambda} u_n \in \operatorname{span} S.$$

($\Leftarrow$) Suppose $v \in \operatorname{span} S$, so

$$v = \lambda_1 u_1 + \cdots + \lambda_n u_n$$

for some $\lambda_i \in F$, $u_i \in S$. Since $v \notin S$,

$$0 = \lambda_1 u_1 + \cdots + \lambda_n u_n - v$$

is a nontrivial linear rel'n on $S \cup \{v\}$ so $S \cup \{v\}$ is lin dep. $\square$

E.g. (1) Let $V = (\mathbb{Z}/3\mathbb{Z})^3$, a $\mathbb{Z}/3\mathbb{Z}$-vs.
Note $|V| = 3^3 = 27$. Now

$$W = \{(x_1, x_2, x_3) \in V \mid x_1 + x_2 + x_3 = 0\}$$

is a subspace. We have

$$W = \{(-x_2 - x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{Z}/3\mathbb{Z}\}$$

$$= \{(0,0,0), (2,1,0), (1,2,0), (2,0,1), (1,1,1), (0,2,1), (1,0,2), (0,1,2), (2,2,2)\}$$

Let's find a lin ind generating set for W, starting with $v_1 = (2,1,0)$. Have

$$\operatorname{span}\{v_1\} = \{(0,0,0), (2,1,0), (1,2,0)\}.$$

Take $v_2 = (1,1,1) \in V \setminus \operatorname{span}\{v_1\}$. By the prop, $S = \{v_1, v_2\}$ is lin ind. Claim $\operatorname{span} S = W$.

Since $S \subseteq W$, know $\operatorname{span} S \subseteq W$. By the thm, every elt of $\operatorname{span} S$ takes the form

$$\lambda v_1 + \mu v_2$$

for unique $\lambda, \mu \in \mathbb{Z}/3\mathbb{Z}$. Thus $|\operatorname{span} S| = 3^2 = 9$.

Since $\operatorname{span} S \subseteq W$ and $|\operatorname{span} S| = 9 = |W|$, must have $\operatorname{span} S = W$.

(2) Let $V = (\mathbb{Z}/5\mathbb{Z})^2$, a $\mathbb{Z}/5\mathbb{Z}$-vs. Then

$$\operatorname{span}\{(1,3)\} = \{0\cdot(1,3) = (0,0),\ 1\cdot(1,3) = (1,3),\ 2\cdot(1,3) = (2,1),\ 3\cdot(1,3) = (3,4),\ 4\cdot(1,3) = (4,2)\}.$$

Here's a picture: