Day 6

Learning Goals

- linear dependence
- linear independence

Future goal Determine "minimal" generating sets of vector spaces/subspaces. To do so, we need a good notion of when a generating set is "insufficient."

Fix field F and F-vs V.

Defn A set $S \subseteq V$ is linearly dependent if $\exists$ distinct $u_1, \ldots, u_n \in S$ and $\lambda_1, \ldots, \lambda_n \in F$ not all 0 s.t. $\lambda_1 u_1 + \cdots + \lambda_n u_n \overset{\circledast}{=} 0$.

Call $\circledast$ a (nontrivial) linear relation among $u_1, \ldots, u_n$. The trivial linear relation is

$$0u_1 + \cdots + 0u_n = 0.$$

E.g. (0) $\varnothing$ is not linearly dependent

(1) If $0 \in S$, then S is linearly dependent: $1 \cdot 0 = 0$.

(2) Let's determine whether

$S = \{(1,-1,0), (-1,0,2), (-5,3,4)\} \subseteq \mathbb{R}^3$ is linearly dependent. Need to find all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ s.t.

$$\lambda_1(1,-1,0) + \lambda_2(-1,0,2) + \lambda_3(-5,3,4) = (0,0,0).$$

I.e.,

$$\begin{aligned} \lambda_1 - \lambda_2 - 5\lambda_3 &= 0 \\ -\lambda_1 \qquad + 3\lambda_3 &= 0 \\ 2\lambda_2 + 4\lambda_3 &= 0 \end{aligned}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & -5 & 0 \\ -1 & 0 & 3 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right) \overset{\text{G-J red'n}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So sol'ns $\lambda_1 - 3\lambda_3 = 0 \Rightarrow \lambda_1 = 3\lambda_3$

$\lambda_2 + 2\lambda_3 = 0 \Rightarrow \lambda_2 = -2\lambda_3$

i.e. $\{(3\lambda_3, -2\lambda_3, \lambda_3) \mid \lambda_3 \in \mathbb{R}\}$

$= \text{span}\{(3,-2,1)\}$ which is infinite.

Taking, e.g., $\lambda_3 = 1$, get the nontrivial linear relation

$$3(1,-1,-5)-2(-1,0,3)+(0,2,4)=(0,0,0)$$

so S is linearly dependent.

Prop A subset $S \subseteq V$ is linearly dependent iff $\exists\, v \in S$ s.t. v is a linear combination of vectors in $S \setminus \{v\}$, i.e., $v \in \operatorname{span}(S \setminus \{v\})$.

Pf First note $S \neq \emptyset$ since $\emptyset$ is not linearly dependent and $\nexists\, v \in \emptyset$.

($\Rightarrow$) Suppose $\lambda_1 u_1 + \cdots + \lambda_n u_n = 0$ is a nontrivial linear relation with $u_1, \ldots, u_n$ distinct elts of S.

WLOG, may assume $\lambda_1 \neq 0$. Then

'without loss of generality'

$$u_1 = -\frac{\lambda_2}{\lambda_1} u_2 - \frac{\lambda_3}{\lambda_1} u_3 - \cdots - \frac{\lambda_n}{\lambda_1} u_n$$

so we may take $v = u_1$.

(Does this work for $S = \{0\}$?

Yes: $0 \in \operatorname{span}(\{0\} \setminus \{0\}) = \operatorname{span} \emptyset = \{0\}$.)

($\Leftarrow$) If $v = \lambda_1 u_1 + \cdots + \lambda_n u_n$ with $\lambda_i \in F$, $u_i \in S \setminus \{v\}$ distinct, then $0 = \lambda_1 u_1 + \cdots + \lambda_n u_n - v$ is a nontrivial linear rel'n amongst elts of S. $\square$

Defn A set $S \subseteq V$ is *linearly independent* if it is not linearly dependent. This means that, for all distinct $u_1, \ldots, u_n \in S$,

$$\lambda_1 u_1 + \cdots + \lambda_n u_n = 0 \Rightarrow \lambda_1, \ldots, \lambda_n = 0.$$

E.g. (0) $\emptyset \subseteq V$ is linearly independent.

(1) $\forall u \in V \setminus \{0\}$, $\{u\}$ is linearly independent:

$$\lambda u = 0 \text{ for } \lambda \neq 0 \Rightarrow u = \tfrac{1}{\lambda} 0 = 0 \quad ♀$$

(2) The set $S = \{(1,-1,0), (-1,0,2), (0,1,1)\} \subseteq \mathbb{R}^3$ is linearly independent: The equation

$$\lambda_1 (1,-1,0) + \lambda_2 (-1,0,2) + \lambda_3 (0,1,1) = 0$$

is equivalent to

$$\begin{aligned} \lambda_1 - \lambda_2 &= 0 \\ -\lambda_1 + \lambda_3 &= 0 \\ 2\lambda_2 + \lambda_3 &= 0 \end{aligned}$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) \overset{G-J}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

so unique sol'n is $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$.

(3) The set $\{1, x, x^2, \dots\} \subseteq F[x]$ is linearly idependent.

Prop For $S \subseteq T \subseteq V$, S lin dep $\Rightarrow$ T lin dep and T lin ind $\Rightarrow$ S lin ind.

Pf TPS. $\square$