Day 5 Learning Goals:

- linear combinations
- span
- more subspaces

TPS For $W_1, W_2 \leq V$, are $W_1 \cap W_2$, $W_1 \cup W_2$ subspaces?

Fix V an F-vs.

Defn For $S \subseteq V$ nonempty, a linear combination of vectors in S is a vector of the form

$$\sum_{i=1}^{n} \lambda_i u_i = \lambda_1 u_1 + \cdots + \lambda_n u_n$$

for some $\lambda_i \in F$, $u_i \in S$, $n \in \mathbb{N}$.

E.g. Is $(-1,4)$ a linear combination of vectors in $\{(3,2), (2,-1)\} \subseteq \mathbb{R}^2$?

Looking for $a, b \in \mathbb{R}$ s.t. $a(3,2) + b(2,-1) = (-1,4)$

i.e.
$$\begin{aligned} 3a + 2b &= -1 \\ 2a - b &= 4 \end{aligned} \rightsquigarrow \left(\begin{array}{cc|c} 3 & 2 & -1 \\ 2 & -1 & 4 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right)$$

so $a = 1$, $b = -2$ works. Indeed,

$$1 \cdot (3,2) - 2(2,-1) = (3-4, 2+2) = (-1,4) \checkmark$$

Defn For $\emptyset \neq S \subseteq V$, the **span** of S is

$$\operatorname{span}(S) := \{\text{linear combinations of vectors in } S\}.$$

By convention, $\operatorname{span}(\emptyset) := \{0\}$.

the "empty linear combo".

E.g. • $\operatorname{span}(\{v\}) = \{\lambda v \mid \lambda \in F\}$

• $\operatorname{span}_{\mathbb{R}}(\{(1,1)\}) = \{(a,a) \mid a \in \mathbb{R}\}$

• $\operatorname{span}_{\mathbb{R}}\{(1,0,0), (0,1,0)\}$

$$= \{a(1,0,0) + b(0,1,0) \mid a, b \in \mathbb{R}\}$$

$$= \{(a,b,0) \mid a, b \in \mathbb{R}\}$$

! Spanning sets are not unique.

$$\operatorname{span}\{(1,0,0), (0,1,0)\}$$

$$= \operatorname{span}\{(1,0,0), (0,2,0)\}$$

$$= \operatorname{span}\{(1,0,0), (0,1,0), (2,3,0)\}.$$

Prop Let S be a subset of V. Then

(1) $\operatorname{span}(S) \leq V$,

(2) If $S \subseteq W \leq V$ then $\operatorname{span}(S) \leq W$.

(3) Every subspace of V is the span of some subset of V.

Pf (1) If $S = \emptyset$, then $\operatorname{span} S = \{0\} \leq V$. If $S \neq \emptyset$ then $\exists u \in S$ so $0 \cdot u = 0 \in \operatorname{span} S$. Given $\sum \lambda_i u_i, \sum \mu_j w_j \in \operatorname{span} S$ where $\lambda_i, \mu_j \in F$, $u_i, w_j \in S$, we have

$$\sum \lambda_i u_i + \sum \mu_j w_j \in \operatorname{span} S$$

& $\lambda \sum \lambda_i u_i = \sum (\lambda \lambda_i) u_i \in \operatorname{span} S$,

so $\operatorname{span} S \leq V$.

(2) Being a subspace, W is closed under linear combinations, so $\operatorname{span} S = \{\text{lin combos of elts of } S\}$ $\subseteq \{\text{lin combos of elts of } W\} = W$. By (1), $\operatorname{span} S$ is also a subspace of W.

(3) $W = \operatorname{span} W$ for $W \leq V$. $\square$

Defn $S \subseteq V$ generates $W \leq V$ when $\operatorname{span} S = W$.

E.g. (1) $\{1, x, x^2, x^3, \dots\}$ generates $F[x]$, polynomials in x w/ coeffs in F.

(2) $\{(1,0), (0,1)\}$ generates $\mathbb{R}^2$ — so do many other sets

(3) The i-th standard basis vector for F^n (to be defined...) is

$$e_i := (0, \dots, 0, \underset{\text{i-th position}}{1}, 0, \dots, 0).$$

We have that

$\{e_1, e_2, \dots, e_n\}$ generates F^n.

(4) For S a set and $s \in S$, define the characteristic function of s by

$$\chi_s : S \longrightarrow F$$
$$t \longmapsto \begin{cases} 1 & \text{if } t = s \\ 0 & \text{if } t \neq s. \end{cases}$$

If S is finite, $\{\chi_s \mid s \in S\}$ generates F^S.

TPS (a) Let $S = \{1, 2, c\}$ and $f : S \longrightarrow \mathbb{Q}$ be given by $f(1) = 3$, $f(2) = -1$, $f(c) = \frac{1}{4}$. Express f as a linear combination of χ_1, χ_2, χ_c.

(b) Does $\{\chi_s \mid s \in S\}$ generate F^S when S is infinite?