Day 4

Learning Goals:

- Two operations and eight properties of vector spaces
- Examples
- Subspaces

Fix a field F, e.g. $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}/p\mathbb{Z}, \dots$, but not $\mathbb{Z}$.

Defn A vector space over F is a set V together with operations

$$+ : V \times V \longrightarrow V \qquad \text{(vector addition)}$$

$$\cdot : F \times V \longrightarrow V \qquad \text{(scalar multiplication)}$$

s.t. $\forall u, v, w \in V$, $\lambda, \mu \in F$

(1) [commutativity of +] $u+v = v+u$

(2) [associativity of +] $u+(v+w) = (u+v)+w$

(3) [additive unit] $\exists\, 0 \in V$ s.t. $0+u=u$

(4) [additive inverses] $\exists -u \in V$ s.t. $u+(-u)=0$

(5) [multiplicative unit] $1 \cdot u = u$

(6) [associativity of scalar mult] $\lambda(\mu u) = (\lambda\mu)u$

(7) [distribution of scalar mult over vector addition]

$$\lambda(u+v) = \lambda u + \lambda v$$

(8) [distribution of scalar mult over field addition]

$$(\lambda + \mu)u = \lambda u + \mu u$$

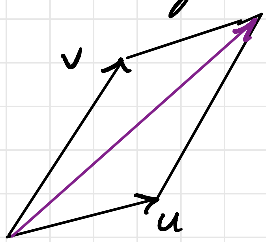
(1) – (4) make $(V, +)$ an $\underbrace{\text{Abelian}}_{=\text{commutative (1)}}$ group.

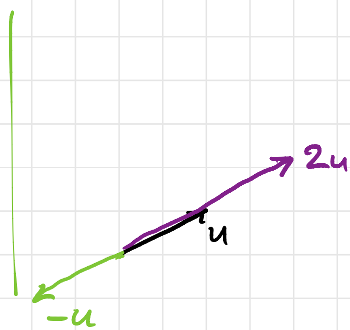
E.g. • For $n \in \mathbb{N}$, $F^n = \underbrace{F \times \cdots \times F}_{n \text{ times}} = \{(x_1, \ldots, x_n) \mid x_i \in F \ \forall i\}$ with operations

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n).$$

• If $F = \mathbb{R}$, $n = 2$, get Euclidean plane with "parallelogram rule" addition and scalar mult'n scaling length:





- If $F = \mathbb{Z}/3\mathbb{Z}$, $n=4$, get computations such as

$$(1,0,2,1) + 2(0,0,1,1) = (1,0,1,0) \in F^4.$$

- If $n=1$, then $F^n = F$, the field itself.

More examples

(1) $\mathbb{C}$ is an $\mathbb{R}$-vector space where if $z = a+bi \in \mathbb{C}$, $a, b, \lambda \in \mathbb{R}$, then $\lambda(a+bi) = (\lambda a) + (\lambda b)i$;

addition is usual complex addition.

(1') If $F \subseteq L$ are both fields, the L is an F-vector space. E.g. $\mathbb{R}$ is a $\mathbb{Q}$-vs. vector space

(2) $\mathrm{Mat}_{m\times n}(F) = \{m \times n \text{ matrices w/ entries in } F\}$

$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \;\middle|\; a_{ij} \in F,\ 1 \le i \le m,\ 1 \le j \le n \right\}$$

Given $A \in \mathrm{Mat}_{m\times n}(F)$, write A_{ij} for its ij-th entry. Then for $A, B \in \mathrm{Mat}_{m\times n}(F)$

$\lambda \in F$, $\quad (A+B)_{ij} = A_{ij} + B_{ij}$

$$(\lambda A)_{ij} = \lambda A_{ij}$$

E.g. $F = \mathbb{Q}$, $m=2$, $n=3$:

$$2\begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 3 \end{pmatrix} - 5\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 8 & 1 \end{pmatrix}.$$

(3) If S is any set, then

$$F^S = \{ f \mid f : S \to F \}$$

is an F-vs where for $f, g \in F^S$, $\lambda \in F$,

$$(f+g)(s) = f(s) + g(s)$$

$$(\lambda f)(s) = \lambda f(s)$$

- When $S = \{1, \dots, n\} =: [n]$, $n \in \mathbb{N}$, $F^{[n]}$ is essentially F^n

depending on your formal def'n of ordered n-tuples, this could be literal equality

- When $S = [m] \times [n]$, F^S is essentially $\mathrm{Mat}_{m \times n}(F)$

- If $S = \mathbb{N}$, then $F^{\mathbb{N}}$ is the F-vs of sequences in F.
- For $F = \mathbb{R}$, might be especially curious about $\mathbb{R}^{\mathbb{R}}$, the vector space of functions $\mathbb{R} \to \mathbb{R}$; $\mathbb{R}^{[0,1]}$ also interesting to analysts.

Subspaces

will define shortly

$\mathbb{R}^{\mathbb{R}}$ has many interesting subspaces.

$$\rotatebox{90}{$\supseteq$}$$

$$C(\mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R} \text{ is continuous} \}$$

$$\rotatebox{90}{$\supseteq$}$$

$$C'(\mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R} \text{ has continuous derivative everywhere} \}$$

$$\rotatebox{90}{$\supseteq$}$$

$$C^{\infty}(\mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R} \text{ has continuous derivatives of all orders} \}$$

$$\rotatebox{90}{$\supseteq$}$$

$$\mathbb{R}[x] = \{ f : \mathbb{R} \to \mathbb{R} \mid f(x) \text{ is a polynomial function} \}$$

Each of these has the nice property that

$$f, g \in W \Rightarrow f + g \in W, \quad \lambda f \in W, \quad 0 \in W$$

In fact, each is a vector space in its own right.

Defn A subset $W \subseteq V$ of an F-vs is a *subspace* of V when W is an F-vs with addition and mult'n inherited from V. In this case, write $W \leq V$.

Prop A subset $W \subseteq V$ is a subspace iff

(1) $0 \in W$

(2) W is closed under addition ($u, v \in W \Rightarrow u+v \in W$)

(3) W is closed under scalar mult'n ($u \in W \Rightarrow \lambda u \in W$)

iff

(1) $0 \in W$

(2') $u, v \in W,\ \lambda \in F \Rightarrow u + \lambda v \in W$.

Pf Two. I Lemma 2.9. $\square$

E.g. $\{(x,0) \mid x \in F\} \leq F^2 \geq \{(0,y) \mid y \in F\}$

but $\{(x,0) \mid x \in F\} \cup \{(0,y) \mid y \in F\}$ is not a subspace: $(1,0) + (0,1) = (1,1)$ is not in the set.