Day 3

Learning Goals:

- define echelon and reduced echelon form
- every system has a unique reduced echelon form (algorithm)
- describe solution sets from reduced echelon form.

leave out the "augmentation"!

Defn A matrix is in echelon form when

(1) all rows of just 0's are at the bottom

(2) the pivot of each nonzero row is strictly to the right of the pivot of the previous row

first nonzero term in the row.
aka leading coefficient

e.g.

$$\begin{pmatrix} 1 & * & * & * & * \\ 0 & 0 & 2 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\left\{ \text{upside down staircase} \right.$

Prop We can put any matrix in echelon form via row operations of type (2) and (3)

Pf by algorithm

(1) Interchange rows to get (any) leftmost pivot in top row

(2) Use row op (3) to get 0's below this pivot

(3) Repeat the previous steps with the matrix obtained by deleting first row & col. $\square$

e.g. System:

$$\begin{aligned} 2x_3 + 6x_4 \quad &= 0 \\ x_1 + 2x_2 + x_3 + 3x_4 \quad &= 1 \\ 2x_1 + 4x_2 + 3x_3 + 9x_4 + x_5 &= 5 \end{aligned}$$

Augmented matrix:

$$\left(\begin{array}{ccccc|c} 0 & 0 & 2 & 6 & 0 & 0 \\ 1 & 2 & 1 & 3 & 0 & 1 \\ 2 & 4 & 3 & 9 & 1 & 5 \end{array}\right)$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 2 & 4 & 3 & 9 & 1 & 5 \end{array}\right)$$

$$\xrightarrow{r_3 \to r_3 - 2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{r_3 \to r_3 - \frac{1}{2}r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

now in echelon form!

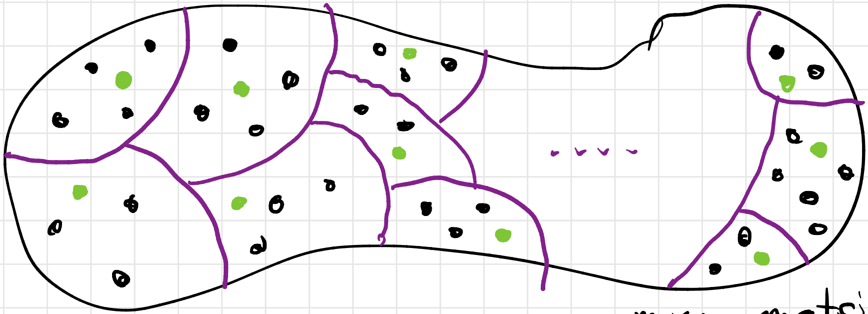
Defn A matrix is in *reduced echelon form* when it is in echelon form, each pivot is 1, and the column above and below the pivot is all 0's.

Thm Every matrix is equivalent (via row ops) to a unique matrix in reduced echelon form.

NB Equivalence of matrices is an equivalence relation, so its equivalence classes partition the set of $m \times n$ matrices. The theorem tells us two things about this:

(1) Row ops *generate* equivalence of matrices.

(2) Each equivalence class has a *unique representative* in reduced echelon form.



$m \times n$ matrices

● = matrices not in REF — reduced echelon form

● = REF representative

◯ = equivalence class

Pf of Thm To produce a REF representative, pass to echelon form via the proposition, then scale rows by $\frac{1}{p_i}$ where p_i is the pivot of the i-th row. Then use row op (3) to clear the column above the pivot.

Uniqueness is more subtle and requires an inductive argument. See Hefferon One.III.2.6. $\square$

e.g. Continuing the previous example, we got echelon form

$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{r_2 \to \frac{1}{2} r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

pivots

$$\xrightarrow{r_1 \to r_1 - r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right).$$

pivot columns

Pivots $\longleftrightarrow$ pivot variables, x_1, x_3, x_5 in this case

Other variables are free variables, x_2, x_4 in this case.

Solution set : $\left\{ (x_1, \dots, x_5) \in F^5 \;\middle|\; \begin{array}{r} x_1 + 2x_2 = 1 \\ x_3 + 3x_4 = 0 \\ x_5 = 3 \end{array} \right\}$

Solve for pivots :
$$x_1 = 1 - 2x_2$$
$$x_3 = -3x_4$$
$$x_5 = 3$$

Can write the solution strictly in terms of free variables :

$$\{(1-2x_2, x_2, -3x_4, x_4, 3) \mid x_2, x_4 \in F\}$$

This is the parametric solution set.

(Sol'ns parametrized by free variables.)

Alternatively, may write solutions in vector form:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} \;\middle|\; x_2, x_4 \in F \right\}.$$

Here using column vectors and, e.g.,

$$x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Addition is componentwise (combining everything recovers parametric form written as a column vector).

e.g. Find all parabolas $f(x) = ax^2 + bx + c$ passing through $(1,4)$ and $(3,6)$.

Soln To pass through $(1,4)$ need

$$4 = a \cdot 1^2 + b \cdot 1 + c$$

$$\Leftrightarrow \quad 4 = a + b + c.$$

To pass through $(3,6)$, need

$$6 = a \cdot 3^2 + b \cdot 3 + c$$

$$\Leftrightarrow \quad 6 = 9a + 3b + c.$$

Solve by Gauss-Jordan reduction:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 9 & 3 & 1 & 6 \end{array}\right) \xrightarrow{r_2 \to r_2 - 9r_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -6 & -8 & -30 \end{array}\right) \text{ EF}$$

$$\xrightarrow{r_2 \to -\frac{1}{6} r_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{4}{3} & 5 \end{array}\right) \xrightarrow{r_1 \to r_1 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & -1 \\ 0 & 1 & \frac{4}{3} & 5 \end{array}\right) \text{ REF}$$

So the original system is equivalent to

$$\begin{aligned} a \quad\quad - \tfrac{1}{3}c &= -1 \\ b + \tfrac{4}{3}c &= 5 \end{aligned}$$

and has solution set

$$\left\{ \left(-1+\tfrac{1}{3}c,\ 5-\tfrac{4}{3}c,\ c\right) \mid c\in F \right\}$$

$$= \left\{ \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1/3 \\ 4/3 \\ 1 \end{pmatrix} \;\middle|\; c\in F \right\}.$$

The parabolas have eq'ns

$$f(x) = \left(-1+\tfrac{1}{3}c\right)x^2 + \left(5-\tfrac{4}{3}c\right)x + c$$

and you can check $f(1)=4$, $f(3)=6$.

(See Sage notebook.)