Day 2

Learning goals:

- Algebra and geometry of systems of linear equations
- Intro to Gaussian elimination / row reduction.

NB Today, $F = \mathbb{R}$.

Example 1 Solve
$$\begin{aligned} 3x + 2y &= 5 \\ 2x - y &= 1. \end{aligned}$$

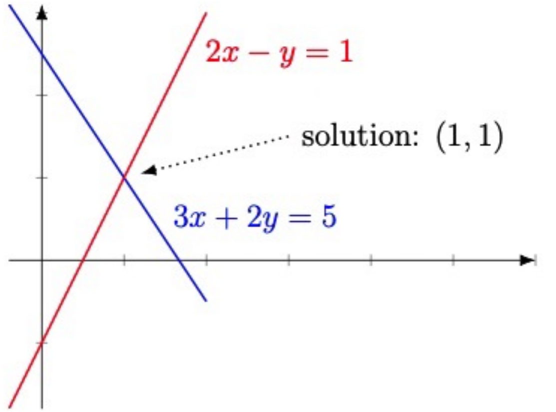
Sol'n Eliminate variables. Multiply second eq'n by 2 then add eq'ns to eliminate y:

$$\begin{array}{rrcl} & 3x + 2y & = & 5 \\ + & 4x - 2y & = & 2 \\ \hline & 7x & = & 7 \end{array} \implies x = 1.$$

Substitute $x = 1$ into first eq'n to get
$$3 + 2y = 5 \implies y = 1.$$

So the unique sol'n is $x = y = 1$. $\square$

Geometric picture:



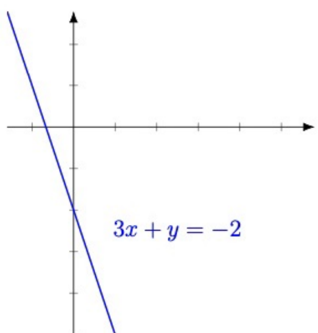
Think-Pair-Share (TPS) What vectors are perpendicular to the line given by $3x+2y=5$? What about $2x-y=1$? How are these related to the eq'ns?

Example 2 System $-9x - 3y = 6$ (1)

$3x + y = -2$. (2)

Since (1) is a scalar multiple of (2), they have the same sol'ns: $\{(x,y) \mid y = -3x-2\}$.

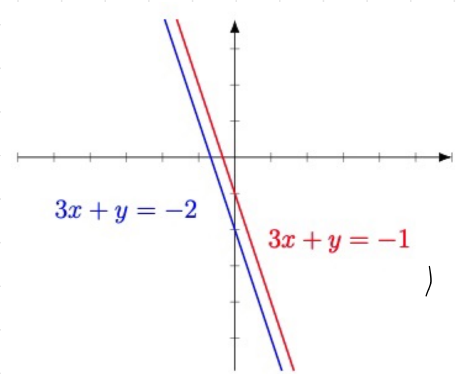
Geometry:



Example 3 System: $-9x - 3y = 6$
$3x + y = -1$.

Dividing first eq'n by -3, get $3x + y = -2$.
Thus no $(x, y) \in \mathbb{R}^2$ satisfies both eq'ns.

Geometry:



parallel lines (same perpendicular vectors too!)

Example 4 (Gaussian elimination)

System:
$$
\begin{aligned}
x + 2y + z &= 0 \\
x \quad\;\; + z &= 4 \\
x + y + 2z &= 1.
\end{aligned}
$$

Idea: Replace this system with a system having the same set of solutions which, moreover, are evident from the form of the system.

same set of sol'ns

Create "equivalent systems" via row operations:

(1) Multiply an eq'n by a nonzero scalar.

(2) Swap two eq'ns.

(3) Add a multiple of one eq'n to another.

Pause Convince yourself that the solution set is invariant under these operations.

In implementing these operations, it's nice to have a shorthand for systems:

$$\begin{aligned} x + 2y + z &= 0 \\ x \quad\;\; + z &= 4 \\ x + y + 2z &= 1 \end{aligned} \quad \rightsquigarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 2 & 1 \end{array}\right)$$

augmented matrix: just record the coefficients!

Let r_i = i-th row of the augmented matrix.

Observe: eliminate x from 2,3

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 2 & 1 \end{array}\right) \xrightarrow[r_3 \to r_3 - r_1]{r_2 \to r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & 0 & 4 \\ 0 & -1 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{r_2 \to -\frac{1}{2} r_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & -1 & 1 & 1 \end{array}\right) \xrightarrow{r_3 \to r_3 + r_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$y = -2$

eliminate y from 3

$z = -1$

$$\xrightarrow{r_1 \to -2r_2 + r_1} \left(\begin{array}{cccc} 1 & 0 & 1 & -4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{r_1 \to -r_3 + r_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

eliminate y from 1

eliminate z from 1

We learn that the unique solution is $(x, y, z) = (5, -2, -1)$.

Pause Check that this is a sol'n to the original system!

Example 5 System:

$$\begin{aligned} x + 2y + z &= 0 \\ x \quad\;\; + z &= 4 \\ x + y + z &= 1 \end{aligned}$$

Same as previous, but this was $2z$.

Performing a similar sequence of row op'ns, get

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

corresponding to

$$\begin{aligned} x &= 4 \\ y &= -2 \\ 0 &= -1 \end{aligned}$$

Upshot: <u>NO</u> sol'ns to this system

(minimalist skull and crossbones := contradiction)

Example 6 Another small modification:

$$\begin{aligned} x + 2y + z &= 0 \\ x + z &= 4 \\ x + y + z &= 2 \end{aligned}$$

same as previous but $1 \rightsquigarrow 2$.

Gaussian reduction:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 1 & 2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

corresponding to $x+z=4$

$$y=-2$$

$$0=0$$

Thus get infinite solution set

$$\{(x,y,z)\in\mathbb{R}^3 \mid x+z=4,\ y=-2\}$$

$$=\{(x,-2,4-x) \mid x\in\mathbb{R}\},$$

a line in $\mathbb{R}^3$.

NB Each "row reduction" today ended in "reduced echelon form." We will

(a) define this precisely,

(b) prove that the Gaussian reduction algorithm always results in this form,

(c) understand solution sets from this form.